MATH 234 BL1 LECTURE 01 NOTES

1. Why learn Calculus?

Calculus is a set of tools and techniques useful for

(1) Maximizing profit and revenue, minimizing cost;
(2) Understanding market states and trends;
(3) Efficiently using resources;
(4) Understanding and predicting dynamic financial situations such as income streams and types of interest;
(5) Understanding dynamic physical scenarios, such as variable velocities, population growth and decay, spreads of diseases and rumors;
(6) Computing physical properties of irregular shapes (volumes, masses, etc)

and literally thousands of other practical and theoretical problems.

2. What is Calculus?

Calculus is a set of techniques for understanding dynamic change, centered around the **derivative** and the **integral**. The derivative allows us to measure the rates of change of dynamically changing quantities and allows us to optimize and analyze quantites that are expressible as mathematical functions. The integral allows us to add up the cumulative effect of many small changes to make aggregate sense of them. These two techniques are unified in the *fundamental theorem of calculus*.

Our first task of the semester is to understand these tools and how to apply them. As is traditional, we start by approaching derivatives through the concept of functions and limits.

3. Functions

To be able to apply calculus to a physical situation, we generally need to express that situation in terms of mathematical functions. Before we discuss functions we need to breifly discuss interval notation.

3.1. **Interval Notation.** For our purposes, there are two basic types of intervals: open and closed. **Closed intervals**, such as the interval $[0, 1]$ indicate the set that contains every real number x that lies between 0 and 1: all x such that $0 \leq 1$. In general, $[a, b]$ is the set of all real x such that $a \leq x \leq b$. **Open intervals** are very similar, but do not include the endpoints. For example, $(0, 1)$ is the collection of all real numbers larger than 0 but smaller than 1 – all real x such that $0 < x < 1$. (Notice that the open interval $(0, 1)$ has no smallest element, but the interval $[0, 1]$ does

– it is 0. Intervals can also be **half-open**, such as $[2, 4)$ which means all real x such that $2 \leq x < 4$. To denote the set of all real numbers, we use the notation $(-\infty, \infty)$ or the symbol $\mathbb{R}$. Endpoints that are infinite are always open.

Functions are rules that assign to each value in a set called the **domain** a value in a set called the **range**. The **natural domain** of a function is the set of all possible input values. We generally always use the natural domain as it is the largest possible domain unless otherwise stated. To find the natural domain, just make sure that you are not taking even roots of negative numbers or dividing by zero. This will suffice for most functions that we encounter, and I will indicate otherwise when exceptions occur.

Examples 1. Consider the following functions and find the domain and range for each.

(1) A line: $f(x) = 2x + 4$. Domain: All real numbers $== (-\infty, \infty)$, Range: $(-\infty, \infty)$

(2) A parabola: $g(x) = x^2$. Domain: $(-\infty, \infty)$, Range: $(0, \infty)$

(3) Square roots: $h(x) = \sqrt{x}$. Domain: $[0, \infty)$, Range: $[0, \infty)$

(4) A hyperbola: $H(x) = \dfrac{1}{x}$. Domain (and Range): All reals except $x = 0 == x \neq 0 == (-\infty, 0) \cup (0, \infty)$.

3.2. **Cost, Revenue, and Profit.** Suppose that a company or firm produces units of a particular good. The cost of producting x units is given by a function $C(x)$, called the **cost function**. If we want to know the cost of the (say) 5th unit, it is the cost of the first 5 units minus the cost of the first four units: $C(5) - C(4)$. The price *per unit* of one unit is given by a **price function** $p(x)$, which is determined by a variety of factors, such as market demand, and so is often called the **demand function**. The amount of revenue generated by selling x units is given by the **revenue function** $R(x)$.

If the company produces and sells x units of the good, the company receives $R(x) = xp(x)$ dollars of revenue (price per unit $\times$ number of units sold). Profit is revenue minus cost, so we define the **profit function** $P(x) = R(x) - C(x)$. (Note the capital P.)

If we solve the equation $P(x) = 0$ or equivalently the equation $R(x) = C(x)$, we find what are called **breakeven points** – points where revenue and cost are equal. Assuming that profit increases as a function of units sold, this is the point at which the company *begins to earn profit.*

Let's consider a specific example.

Example 2. Suppose a factory has \$100 of fixed costs of operation (rent, electricity) and can produce sprockets at a cost of \$5 each. If consumers are willing to pay \$10 for each sprocket, how many units must the factory product to be profitable?

Solution: Since the units cost \$5 each and the factory cost \$100 to run, the cost function is given by $C(x) = 100 + 5x$ (= fixed costs + production costs). The price function is $p(x) = 10$, so the revenue function is $R(x) = xp(x) = 10x$, and so the profit function is $P(x) = R(x) - C(x) = 10x - (100 + 5x) = 5x - 100$.

The breakeven point is given as the solution of $P(x) = 0$: $5x - 100 = 0 \Rightarrow x = 20$. If the factory produces and sells more than 20 sprockets, the factory will be profitable.

3.3. **Operations on Functions; Compositions of Functions.** Functions behave like numbers in many respects. Given two functions f and g, we can form several new functions:

(1) Sum: $(f + g)(x) = f(x) + g(x)$

(2) Difference: $(f - g)(x) = f(x) - g(x)$

(3) Product: $(fg) = f(x) * g(x)$

(4) Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ if $g(x) \neq 0$

Notice that we used the difference of two function above to define profit.

We can also **compose** functions by substituting one function into another:

$$(f \circ g)(x) = f(g(x)).$$

Examples 3. Let $f(x) = \sqrt{x + 1}$ and $g(x) = x^2 + 2$. Then we have

(1) $f(g(x)) = \sqrt{(g(x))^2 + 1} = \sqrt{(x^2 + 2) + 1} = \sqrt{x^2 + 3}$

(2) $g(f(x)) = (f(x))^2 + 2 = (\sqrt{x + 1})^2 + 2 = x + 1 + 2 = x + 3$

Notice that the order matters when composing functions!

3.4. **Recognizing functions as compositions.** It will be *extremely* important for us to recognize functions as compositions of other functions (when we used the ubiquitous chain rule for derivatives). For example, the function $h(x) = \sqrt{x + 1}$ is a composition of the function $f(x) = \sqrt{x}$ and the function $g(x) = x + 1 : h(x) = f(g(x))$. For a more complicated example, consider the following.

Example 4. Write the function $h(x) = (3x + 5)^2 + \frac{1}{3x+5}$ as a composition of functions.

Solution: Notice that the *inner* function seems to be $3x + 5$. Let's call that g, so $g(x) = 3x + 5$. Now, $h(x)$ appears to square g and then to divide by g, so we see that if $h(x) = f(g(x))$ then $f(x) = x^2 + \frac{1}{x}$. If you don't believe this, expand out $f(g(x))$ to see that it equals $h(x)$.

4. Homework

Read Section 1.1., examples in particular. Look at the practice problems on mathzone. Additional practice: Section 1.1 – 20, 21, 31, 49, 66, 67 (page 9).