

MATH 234 BL1 LECTURE 04 NOTES

SECTION 1.5: LIMITS / SECTION 2.1 DERIVATIVES

1. MORE ON LIMITS AT INFINITY

The basis for understanding limits at infinity starts with the following:

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0,$$

as we can see from the graph of the function. By the power rule for limits, we also have that

$$\lim_{x \rightarrow \infty} \frac{1}{x^p} = 0,$$

for any *constant* value of $p > 0$.

It is also important to note that:

$$\lim_{x \rightarrow \infty} x = \infty,$$

and more generally

$$\lim_{x \rightarrow \infty} x^p = \infty,$$

for any *constant* value of $p > 0$.

From these limits, we can understand all limits of rational functions of the form

Example 1. Evaluate the limit

$$\lim_{x \rightarrow \infty} \frac{x + 3}{x^2 + 2x}.$$

Solution: To find the limit, since the degree of the denominator (the highest power present, in this case 2) is greater than the degree of the numerator (in this case 1), we divide the fraction through by x^2 , the largest degree term in the denominator. We have

$$\lim_{x \rightarrow \infty} \frac{x + 3}{x^2 + 2x} = \lim_{x \rightarrow \infty} \frac{x + 3}{x^2 + 2x} \times \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{3}{x^2}}{1 + \frac{2}{x}}.$$

Now, we apply the algebraic rules for limits (sums and quotients) to obtain

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{3}{x^2}}{1 + \frac{2}{x}} = \frac{\lim_{x \rightarrow \infty} (\frac{1}{x}) + 3 \lim_{x \rightarrow \infty} (\frac{1}{x^2})}{1 + 2 \lim_{x \rightarrow \infty} (\frac{1}{x})} = \frac{0 + 3(0)}{1 + 2(0)} = 0.$$

Example 2. Evaluate the limit

$$\lim_{x \rightarrow \infty} \frac{2x^2 + 3}{3x^2 + 2x}.$$

Solution: Applying the same technique as the above example,

$$\lim_{x \rightarrow \infty} \frac{x + 3}{x^2 + 2x} = \lim_{x \rightarrow \infty} \frac{2 + \frac{3}{x^2}}{3 + \frac{2}{x}} = \frac{2 + 0}{3 + 0} = \frac{2}{3}.$$

In general, the rule is:

$$\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if degree } p < \text{degree } q \\ \pm\infty & \text{if degree } p > \text{degree } q \\ \frac{p_0}{q_0} & \text{if degree } p = \text{degree } q \end{cases},$$

where p_0 and q_0 are the leading coefficients of $p(x)$ and $q(x)$, respectively.

2. DERIVATIVES

We've seen that limits can allow us to investigate the behavior of functions near points where a function is not defined (i.e. points not in the natural domain of the function).

Recall that the formula for the slope of the secant line of a function between the point $(x, f(x))$ and a nearby point $(x + h, f(x + h))$ is

$$m = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}.$$

Notice that since there is a lone h in the denominator, this function is not defined when $h = 0$; in other words, $h = 0$ is not in the domain of this function viewed as a function of the variable h .

But we know that as $h \rightarrow 0$ that this formula gives the slope of the tangent line at the point $(x, f(x))$ and we know that the slope of the tangent line measures the rate of change (which we want to know!), so let's define the slope of the tangent line to be:

$$m = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}.$$

After we take this limit (assuming it exists), the variable h will disappear, and we'll know something about the point $(x, f(x))$.

Example 3. Find the slope of the tangent line of the function $f(x) = \frac{1}{x}$ when $x = 1$.

Solution: When $x = 1$, $f(x) = \frac{1}{x} = 1$, so we are looking for the slope of the tangent line at the point $(1, 1)$. By definition, this is the limit

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{1+h}\right) - \frac{1}{1}}{h}.\end{aligned}$$

That's not very pretty or easy to deal with. Let's do some algebra:

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\frac{1}{1+h} - \frac{1}{1}}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{1+h} - \frac{1+h}{1+h}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1-(1+h)}{1+h}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{-h}{1+h}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{(1+h)h} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(1+h)}.\end{aligned}$$

Finally, we can let $h = 0$ and obtain that the limit is $\frac{-1}{1+0} = -1$.

That was a lot of work. We certainly don't want to do that for every point individually, so let's do it for every point x as the same time.

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{x+h}\right) - \frac{1}{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{x(x+h)h} \\ &= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} \\ &= \frac{-1}{x(x+0)} = -\frac{1}{x^2}.\end{aligned}$$

That was even more work! But now we've got a new function, call it $f'(x) = -\frac{1}{x^2}$, that tells us the slope of the tangent line at any point (except $x = 0$). The slope at $x = 2$? It's $f'(2) = -\frac{1}{2^2} = -1/4$. At $x = -1/2$? The slope is $f'(-1/2) = -1/(-1/2)^2 = 4$. Since this new function f' contains

so much information, let's give it a name – the **derivative** of f , and let's define it for every function ever:

Definition 1 (The Derivative). For any function f , let the function f' be defined as

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

whenever the limit exists, and call it the **derivative** of f .

Let's calculate the derivative for some other functions.

Example 4. Find the derivative of $f(x) = x^2$.

Solution: Start from the definition:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2x+h)h}{h} \\ &= \lim_{h \rightarrow 0} 2x + h = 2x + 0 = 2x, \end{aligned}$$

So the derivative is $f'(x) = 2x$. We can find the slope at $x = 1$ (as we did in the last lecture), this time getting $f'(1) = 2(1) = 2$.

Well, the derivative method is more general and will save us some work, but it is tedious to calculate these limits all the time. Thankfully, if you compute enough of them, you start to see patterns. For example, consider the following:

function	derivative	why?
$1 = 1x^0$	0	it's a horizontal line, so it has slope 0
$x = x^1$	$1 = 1x^0$	it's a line of slope 1
x^2	$2x = 2x^1$	see above
x^3	$3x^2$	from the definition (try it!)

The pattern goes like this: if $f(x) = x^n$, then $f'(x) = nx^{n-1}$, at least for positive n . There's a better way to say it, though – let the symbol $\frac{d}{dx}$ mean "take the derivative of" so that we can write

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

This rule is called the **power rule** and is extremely useful.

What about our friend $f(x) = \frac{1}{x}$ from above? We write

$$\frac{d}{dx} \left[\frac{1}{x} \right] = -\frac{1}{x^2}.$$

Hey! If we write $\frac{1}{x} = 1x^{-1}$ and apply the power rule, we get $-1x^{-1-1} = -1x^{-2} = -\frac{1}{x^2}$. It looks like the power rule works for negative values of n as well!

Read through the extra examples in section 1.5 of the book.