MATH 234 BL1 LECTURE 05 NOTES

SECTION 1.6: ONE-SIDED LIMITS AND CONTINUITY

Previously we discussed how to use limits to calculate tangent lines and instantaneous rates of change. When can there fail to be a tangent line at a point on a function? Consider the function $f(x) = x^{2/3}$. The graph has a sharp cusp at the point $x = 0$ which makes it unclear how to actually draw in a tangent line. To understand this situtation and rates of change in general, we need to study limits in a bit more depth.

When we first starting talking about functions and limits, we mentioned for *nice* functions, such as polynomials, we can calculate limits just by plugging into the function:

$$\lim_{x\to c} p(x) = p(c)$$

Today we'll investigate functions that have this property and learn how to tell if a limit exists.

1. Continuity

Intuitively, a *continuous function* is a function that can be drawn without lifting your pen – it has no holes, jumps, or asymptotes, referred to as **discontinuities**. More precisely, a function is continuous at a point if we can approach it from both sides from some close enough distance that we do not encounter any discontinuities. This is the same as the limit of the function existing and having the same value as the function at that point, so let's make the following definition.

Definition 1. A function f is **continuous at a point** c if

(1) the limit $lim_{x\to c} f(x)$ exists, and
(2) the value of the limit agrees with the function, i.e. $lim_{x\to c} f(x) = f(c)$.

If a function is continuous at every point on an interval, we say it is a **continuous function** (on that interval).

Examples 1. Continuous functions start with the *constant functions* $f(x) = K$, which are just horizontal lines – they are defined everywhere and always take the same value. Functions of the form $f(x) = ax + b$ are also just lines and hence are also continuous (they are just rotations of constant functions), and all polynomials are continuous because they are just sums and products of lines, so by the algebraic rules for limits, they are also continuous.

Rational functions of the form $f(x) = \dfrac{p(x)}{q(x)}$ are continuous as long as $q(x)$ is never zero, because then the function would have an asymptote. For example, $f(x) = \frac{1}{x^2+1}$ is continuous and so is $g(x) = \frac{x}{x^2+1}$, but $h(x) = \frac{1}{x^2-1}$ is not continuous at $x = 1$ or $x = -1$ because it has asymptotes there, and the value of a limit approaching either of those points approaches either ∞ or $-\infty$.

In general, for our purposes you can tell if a function is continuous by looking for discontinuities. If it has none, it's probably continuous. Let's look at the different types of discontinuities.

2. Discontinuities

There are three types of discontinuities that we commonly encounter.

2.1. Removable Discontinuities.

The first type of discontinuity is called a **removable discontinuity**. This occurs when a graph has a hole, such as the function

$$f(x) = \frac{x^2-1}{x-1} = \begin{cases} x+1 & \text{for } x \neq 1 \\ \text{undefined} & \text{for } x = 1 \end{cases}$$

We can *remove* this discontinuity by defining $f(1) := \lim_{x\to 1} f(x) = 2$. Similarly, the function

$$f(x) = \begin{cases} x+1 & \text{for } x \neq 1 \\ 4 & \text{for } x = 1 \end{cases}$$

is defined at $x = 1$ but is not continuous because the value of the limit, $\lim_{x\to 1} f(x) = 2$ is not equal to the value of the function at the point, which was defined to be $f(1) = 4$. We could redefine that point to make the function continuous.

2.2. Jump Discontinuities.

The second type of discontinuity is called a **jump discontinuity**. This occurs when a graph has a sudden gap, such as the *step function*

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

This function has a sudden jump at $x = 0$ and unlike in the previous case, there is no single point that we can redefine to close the gap. The function is continuous at ever point other than zero though, and we can say more: coming into the point $x = 0$, the function always has the value -1, so the *one-sided limit* from the *left* is

$$\lim_{x\to 1^-} f(x) = -1,$$

and the *one-sided limit* from the *right* is

$$\lim_{x\to 1^+} f(x) = 1.$$

The limit does not exist because the value is different depending on which side you approach the point from!

In general, the limit $\lim_{x\to c} f(x)$ exists if and only if the two one-sided limits exist and agree: $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$.

Example 2. For what value of c is the following function continuous?

$$f(x) = \begin{cases} x^2+3 & \text{if } x < 1 \\ 2x+c & \text{if } x > 1 \end{cases}$$

Solution: For this function to be continuous, it has to be glued together the right way. In other words, we want the left-handed limit and the right-handed limit to agree. They are

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2+3 = 1^2+3 = 4,$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x+c = 2(1)+c = 2+c,$$

so we need $4 = 2 + c$, hence $c = 2$.

2.3. **Asymptotes.** The last type of discontinuity is the **vertical asymptote**. This usually occurs when there is a division by zero in the denominator of a rational function *but not if the point is a removable discontinuity.* So, $f(x) = \frac{1}{x-1}$ has an asymptote at $x = 1$ but the function $f(x) = \frac{x^2-1}{x-1}$ does not have an asymptote at $x = 1$ because the discontinuity is removable.

Example 3. Identify and describe the discontinuties of the functions

$$f(x) = \frac{x+2}{x^2+5x+6}.$$

Solution: We can factor as follows:

$$f(x) = \frac{x+2}{x^2+5x+6} = \frac{x+2}{(x+2)(x+3)} = \frac{1}{x+3},$$

so we have a removable discontinuity at $x = -2$ and an asymptote at $x = -3$. The function is continuous everywhere else.

3. The Relationship between Differentiability and Continuity

Recall that the derivative of a function f is a new function f' that gives the slope of the tangent line of the function f at all points x *for which it exists.* In addition to the example above ($x^{2/3}$), another function that fails to have a derivative is the absolute value function:

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

It has a sharp corner at $x = 0$ that also makes the choice of tangent line unclear.

What goes wrong in this case? Recall the limit that defines the derivative:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Let's consider it as h approaches zero from the left and from the right, at the point $x = 0$.

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|-0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1,$$

where we used that $|h| = -h$ for negative h. On the other hand,

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|-0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1,$$

where we used that $|h| = h$ for positive h. Since the limits on the left and right do not agree, the limit does not exist, and so there is no clear tangent line (and slope) that makes sense to draw at this corner point.

Note however that the absolute value function is continuous at zero since

$$\lim_{h\to 0} |h| = 0.$$

Hence, **continuity does not imply differentiability**.

The converse situation is better though: **differentiability implies continuity**. Why? The argument is as follows.

First, we need another form of the derivative. I claim that

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

is another way to find derivatives, which you can prove by setting $c = x+h$. Suppose we have a differentiable function f; then the limit $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$ at the point x, so we just need to show that $\lim_{x\to c} f(x) = f(c)$ to satisfy the definition of continuity. This is the same as $\lim_{x\to c} [f(x) - f(c)] = 0$. Here's the trick to show that:

$$\lim_{x\to c} [f(x)-f(c)] = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}(x-c) = \lim_{x\to c} f'(c)(x-c) = f'(c) \lim_{x\to c} (x-c) = 0,$$

since $\lim_{x\to c} (x-c) = 0$ ($x-c$ is continuous). Hence we have that $\lim_{x\to c} f(x) = f(c)$ and so f is continuous.