MATH 234 BL1 LECTURE 06 NOTES

SECTION 2.2 - 2.3: DIFFERENTIATION TECHNIQUES

We have discussed limits and have seen how to use limits to find the values of the slope of the tangent line of a function at a point. To a given function f we associate a function f' called the *derivative* of f that gives the slope of the tangent line of the function f at the specified point.

We discussed the property of continuity of functions – a function is continuous at a point $x = c$ if the limit $\lim_{x \to c} f(x) = f(c)$; in other words, the limit exists and is equal to the value of the function at that point. Intuitively, a continuous function is one that can be drawn without lifting one's pencil – it has no breaks, jumps, or gaps.

Last time we looked at some functions that for which the limit defining the tangent line does not exist at a particular point, such as the absolute value function $|x|$ at $x = 0$ and the function $f(x) = x^{2/3}$ at $x = 0$. If the limit defining the tangent line exists at a point x of a function f, we say that the function is differentiable at x; if it is differentiable at all points in its natural domain, we simply say that it is differentiable. The absolute value function is differentiable at every point except $x = 0$ (it is continuous at every point though). So we see that even if a function is continuous at a point (or all points) it may fail to be differentiable, but we saw that if a function is differentiable at a point it is also continuous at that point.

We also discussed the power rule for derivatives:

$$\frac{d}{dx}[x^n] = nx^{n-1} \qquad \text{for all integers } n.$$

1. Where We Are Going

Derivatives turn out to be a very useful tool for investigating functions. For example, maximum and minimum values of a function can occur at points where derivative is equal to zero. Why? Consider a ball that has been thrown into the air. If the function $s(t)$ measures its height at time t then the derivative represents the velocity (speed and direction) of the ball at time t since velocity is the rate of change of displacement (height in this case). The maximum occurs when the ball stops moving, i.e. when the velocity is equal to zero. This occurs when the derivative is equal to zero, since the velocity is the derivative.

The height of a ball thrown from an initial height of h_0 with an initial velocity of v_0 is given by $s(t) = -16t^2 + v_0 t + h_0$ in feet (the -16 is due to the strength of gravity). Let's find the maximum height in a specific situation.

To do this, we'll need to know that the derivative of a sum is the sum of the derivatives (and similarly for subtraction); in formulas

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}\left[f(x)\right]+\frac{d}{dx}\left[g(x)\right],$$

or in the prime notation

$$[f(x)+g(x)]'=f'(x)+g'(x).$$

We'll also need to know that constant multiples can be taken outside limits

$$\frac{d}{dx}\left[cf(x)\right]=c\frac{d}{dx}\left[f(x)\right],$$

or in the prime notation

$$[cf(x)]'=cf'(x).$$

Both of these properties follow from the rules for limits (and the fact that the derivative is a special limit).

Example 1. Find the maximum height of a ball whose height is given by the function $s(t) = -16t^2 + 64t + 3$.

Solution: We want to know when the velocity is equal to zero, so we take the derivative and set it equal to zero. The derivative is

$$s'(t)=\frac{d}{dx}\left[-16t^2+8t+3\right]=\frac{d}{dx}[-16t^2]+\frac{d}{dx}[8t]+\frac{d}{dx}[3]=-16(2t)+64=-32t+64.$$

Setting this equal to zero, we find that if $0 = -32t + 64$ then $t = 2$. *This is the time at which the velocity is equal to zero.* The maximum height is the value of the function at this time: $s(2) = -16 * 2^2 + 64 * 2 + 3 = 67$ ft.

For the record, velocity is the derivative of displacement and acceleration is the derivative of velocity (which is equal to the *second* derivative of displacement). We'll see other uses of derivatives soon, but for now we need some new techniques.

Unfortunately the functions we encounter are not always easy to differentiate as the often occur as products and quotients of other functions. We don't want to use the limit definition every time we encounter such a function, so let's learn some more rules to make the process of finding derivatives easier.

2. Techniques of Differentiation

Often functions occur as the product of two other functions, such as $f(x) = (3x + 5) * (x^3 + 5x + 4)$. We could multiply this all out and use the power rule but that is a lot of work and it won't work for more complicated things like $g(x) = (x^2 + 1) * \sqrt{x + 1}$.

You might guess that the derivative of a product is the product of the derivatives, in analogy with the product rule. Consider though that for $x^2 = x * x$ that the derivative of x^2 is $2x$ but the derivative of x is just 1, so the product of the derivatives is $1 * 1 = 1$, so the conjectured rule cannot be correct. The correct rule is the following, known as the **product rule**, which can be shown from the limit definition:

$$\frac{d}{dx}\left[f(x) * g(x)\right] = \frac{d}{dx}\left[f(x)\right] g(x) + f(x)\frac{d}{dx}\left[g(x)\right],$$

or in the prime (') notation

$$[f(x) * g(x)]' = f'(x)g(x) + f(x)g'(x).$$

Check this for $x^2 = x * x$.

Example 2. Find the derivative of $f(x) = (3x + 5) * (x^3 + 5x + 4)$.

Solution: The derivative of $(3x+5)$ is 3 and the derivative of (x^3+5x+4) is $3x^2+5$, so the derivative of f is $f'(x) = 3*(x^3+5x+4)+(3x+5)*(3x^2+5)$.

Similarly, functions often occur as quotients and there is a technique known as the **quotient rule**:

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Example 3. Find the derivative of $h(x) = \dfrac{x^3}{x^2 - 1}$.

Solution: Using the quotient rule,

$$h'(x) = \frac{3x^2 * (x^2 - 1) - x^3 * (2x)}{(x^2 - 1)^2}.$$

Note that the function in the previous example has no derivative when $x = \pm 1$ because it has asymptotes at those points, but has derivative elsewhere.

Example 4. Find the maximum and minimum values of $h(x) = \dfrac{x}{x^2 + 1}$, given that they occur when the derivative is equal to zero.

Solution: Using the quotient rule,

$$h'(x) = \frac{(1)(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

The maximum and minimum values occur when the slope is equal to zero, which is when the numerator is equal to zero: $1 - x^2 = 0$, so $x = \pm 1$. The values of the function at those points are $h(1) = 1/(1^2 + 1) = 1/2$ and $h(-1) = -1/(1^2+1) = -1/2$, which are the maximum and minimum values, respectively.

Please look over the additional examples in your text.