

MATH 234 BL1 LECTURE 08 NOTES

2.6 IMPLICIT DIFFERENTIATION AND RELATED RATES

Last time we discussed the chain rule:

$$[f(g(x))]' = f'(g(x))g'(x)$$

or in Leibnitz notation, if $u = g(x)$ and $h(x) = f(g(x)) = f(u)$

$$\frac{dh}{dx} = \frac{dh}{du} \frac{du}{dx}.$$

The chain rule is more than just a tool for calculating derivatives – it allows us to take derivatives of implicitly defined functions and also to relate rates.

1. IMPLICIT DIFFERENTIATION

The chain rule allows us to take derivatives of implicitly defined functions as well. Consider the circle $x^2 + y^2 = 1$. Unfortunately we cannot solve explicitly for y , so we can't take the derivative in the usual way as $y'(x)$, but what we can do is to assume that y is a function of x even though we have trouble solving for it. The derivative of y with respect to x is

$$\frac{d}{dx}[y] = \frac{dy}{dx}.$$

By the chain rule, the derivative of y^2 is

$$\frac{d}{dx}[y^2] = 2y \frac{d}{dx}[y] = 2y \frac{dy}{dx} = 2yy'.$$

Example 1. Find the derivative of y if $x^2 + y^2 = 1$.

Solution: Take the x derivative $\frac{d}{dx}$ of everything, making sure to use the chain rule for functions of y :

$$\begin{aligned} \frac{d}{dx}[x^2 + y^2] &= \frac{d}{dx}[1] \\ 2x + 2yy' &= 0 \end{aligned}$$

Solving for x we see that $\frac{dy}{dx} = \frac{-x}{y}$. What is the slope of the tangent line at the point $(0, 1)$ at the top of the circle?

$$\left. \frac{dy}{dx} \right|_{(0,1)} = \left. \frac{-x}{y} \right|_{(0,1)} = \frac{-0}{1} = 0,$$

as it should be since the tangent line is horizontal.

To summarize, consider the following:

$$\begin{aligned}\frac{d}{dx}[y] &= \frac{dy}{dx}[y] \\ \frac{d}{dx}[y^2] &= 2y \frac{d}{dx}[y] = 2y \frac{dy}{dx} \\ \frac{d}{dy}[y^2] &= 2y\end{aligned}$$

2. POWER RULE FOR FRACTIONAL EXPONENTS

Incidentally, implicit differentiation is why the power rule works for fractional exponents. If $y = x^{p/q}$ then $y^q = x^p$. Taking the derivative of each side gives $qy^{q-1}y' = px^{p-1}$ and solving for y' we find that

$$y' = \frac{p x^{p-1}}{q y^{q-1}} = \frac{p x^{p-1}}{q x^{\frac{p(q-1)}{q}}} = \frac{p}{q} x^{\frac{p}{q}-1}.$$

3. RELATED RATES

In general, if you have a function of y , say $f(y)$, the derivative with respect to y is

$$\frac{d}{dy}[f(y)] = f'(y)$$

but with respect to the variable x , the chain rule demands that

$$\frac{d}{dx}[f(y)] = f'(y)y' = f'(y)\frac{dy}{dx}.$$

Something analogous occurs when we try to differentiate a function of x , say $g(x) = 3x^2 + 10x$ with respect to another variable, say t :

$$\frac{d}{dt}[3x^2 + 10x] = 6x \frac{dx}{dt} + 10 \frac{dx}{dt}.$$

Why would we do such a thing? Often the variable of interest depends on time or another variable. Using implicit differentiation, we can relate the rates of such things.

Consider the following example.

Example 2. A ladder 10 feet in length rests on a wall. When the base of the ladder is 6 feet away from the wall, the ladder begins to slide downward at 2 feet per second. At what rate does the ladder slide away from the wall?

Solution: First give names to the quantities in question. Let the ladder be x feet away from the wall and y up the wall. Since the ladder is 10 feet long, the Pythagorean theorem says that $x^2 + y^2 = 10^2 = 100$. We are told that the ladder is 6 feet from the wall and that it begins to slide down at 2 feet per second. If the ladder is 6 feet away from the wall, it must be 8 feet

up the wall since $6^2 + 8^2 = 10^2$. When y is equal to 8, y is decreasing at the rate of 2 feet per second, so

$$\frac{dy}{dt}\bigg|_{y=8} = -2 \text{ ft/sec.}$$

The problem asks for the rate of change of x . We obtain this from the equation $x^2 + y^2 = 100$ by taking the derivative with respect to t : if $x^2 + y^2 = 100$ then

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0,$$

and solving for y we see that $\frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}$. Thus,

$$\frac{dx}{dt}\bigg|_{x=6} = -\frac{8}{6} \frac{dy}{dt}\bigg|_{y=8} = -\frac{16}{6} \text{ ft/sec.}$$

A note from the previous problem – when a quantity is increasing it has positive derivative and when decreasing it has a negative derivative. We'll talk more about the meaning of the sign of the derivative soon. Here is another example.

Example 3. An oil tanker hits an iceberg causing a hull rupture. Oil begins leaking outward in a circular spill at a rate of $400 \text{ m}^3/\text{min}$ into a slick that is cylindrical of width 0.5 m . Sometime after the rupture, a helicopter above notices the radius of the slick is 200 m . At what rate is the circumference of the slick increasing? The Coast Guard wants to know how much containment netting to bring.

Solution: First give names to the quantities in question. Let the radius be r , the width $w = 0.5$, and the volume of the slick $V = \pi r^2 w = 0.5\pi r^2$. We know that the volume is increasing at a rate of $400 \text{ m}^3/\text{min}$, so let's find out how fast the radius is increasing. We'll use that to figure out how fast the area and circumference are increasing. We have that

$$400 = \frac{dV}{dt} = 0.5 * 2\pi r \frac{dr}{dt} = 200\pi \frac{dr}{dt},$$

since $r = 200$, so $\frac{dr}{dt} = 400/200\pi = 2/\pi$. Now the circumference is given by $c = 2\pi r$, so $\frac{dc}{dt} = 2\pi \frac{dr}{dt} = 2\pi * \frac{2}{\pi} = 4 \text{ ft/min}$.

At the current rate of spreading, the Coast Guard needs an additional 4 feet of netting for each minute that passes.