# MATH 234 BL1 LECTURE 12 NOTES

## 3.4 MARGINAL ANALYSIS AND ABSOLUTE EXTREMA

0.1. **Maximum Profit and Minimum Average Cost.** Recall that we defined the profit function as the difference between revenue and cost:

$$P(x) = R(x) - C(x).$$

Let's also define the *average cost function* $A(x)$ to be

$$A(x) = \frac{C(x)}{x},$$

which gives the average price per unit of producing $x$ units of a commodity. Cost is usually written as a sum of fixed costs (such as rent) and the prodcut of the per-unit costs (such as components or ingredients) times the number of units produced. The average cost function spreads out the fixed costs over each unit.

We can use the first an second derivative tests to derive conditions in which the profit is maximized and the minimum average cost is minimized. These are called *marginal analysis criteria* for maximum profit and minimum average cost.

Let's try to maximize the profit. First we need to have a critical point of the profit function, so we need

$$0 = P'(x) = R'(x) - C'(x),$$

so we must have that $R'(x) = C'(x)$. We want this to be a maximum, so we use the second derivative test to decide if we have a maximum or minimum. We want $P''(x) < 0$, so we need

$$R''(x) - C''(x) = P''(x) < 0,$$

and so $R''(x) < C''(x)$.

**Example 1.** Given the functions $R(x) = 49x - x^2$ and $C(x) = \frac{1}{8}x^2 + 4x + 200$, find the maximum possible profit.

**Solution**: First find critical points where $R'(x) = C'(x)$. $R'(x) = 49 - 2x$ and $C'(x) = \frac{1}{4}x + 4$, and setting equal and solving gives $x = 20$. Let's check if the second criterion is fulfilled by taking the second derivatives. $R''(x) = -2$ and $C''(x) = \frac{1}{4}$, so $R''(20) < C''(20)$ and we have a maximum.

Similarly, we can find the minium average cost. First let's find critical points (using the quotient rule):

$$0 = A'(x) = \frac{d}{dx}\left[\frac{C(x)}{x}\right] = \frac{C'(x)x - C(x)}{x^2},$$

and solving yields $0 = C'(x)x - C(x) \Rightarrow C'(x) = \frac{C(x)}{x} = A(x)$, so we have critical point sof $A(x)$ when $A(x) = C'(x)$. This is basically always a minimum: using the quotient rule we have that

$$A''(x) = \frac{[C''(x)x + C'(x) - C'(x)] - 2x(C'(x)x - C(x))}{x^4} = \frac{C''(x) - 2x^2C'(x) - 2xC(x)}{x^4},$$

which will be greater than zero if $C''(x) > 0$, a usually reasonable assumption about the cost function of a production commodity.

**Example 2.** Given the function $C(x) = \frac{1}{8}x^2 + 4x + 200$, find the minimum average cost.

**Solution**: First we compute $C'(x) = \frac{1}{4}x + 4$ (as above) and set this equal to $A(x) = \frac{C(x)}{x}$: $\frac{1}{4}x + 4 = \frac{\frac{1}{8}x^2+4x+200}{x} \Rightarrow 2x^2 + 32x = x^2 + 32x + 1600$, which gives that $x^2 = 1600$ and so $x = 40$. Note that this is different from the point as which maximum cost occurs.

## 1. Absolute Extrema

In sections 3.1 and 3.2 we discussed relative (or local) extrema, which are points on a function that are maximal or minimal in a small interval and correspond to critical points. Since the derivative measures local properties, it is unable to determine **absolute extrema** – the largest and smallest values of a function on a particular domain. Absolute extrema may occur at critical point or the may not exist at all. Let's begin with a definition.

**Definition 1.** Let $f$ be a function defined on an interval $I$ which contains the point $x = c$. We say that $f(c)$ is the *absolute maximum* of $f$ on $I$ if $f(c) \geq f(x)$ for all $x$ in $I$ and that $f(c)$ is the *absolute minimum* of $f$ on $I$ if $f(c) \leq f(x)$ for all $x$ in $I$.

The difference between relative extrema and absolute (or global) extrema is the intervals in question: local extrema require only a (perhaps very) small interval wherease absolute extrema are defined on a large interval, perhaps even $(-\infty, \infty)$.

Notice that a function need not have an absolute maximum even if it gets arbitrarily large. For example, the function $f(x) = x$ has no global maximum because for any point $x = c$ that one tries to declare the maximum, there is a point that is larger, for instance $c + 1$.

Usually we discuss absolute extrema on closed intervals and on the natural domain of a function.

## 2. Optimization on a Closed Interval

To find the absolute extrema of a function $f$ a closed interval $[a, b]$, we have to check the critical points of $f$ and the endpoints of the interval.

**Example 3.** Find the absolute extrema of

$$f(x) = x + \frac{1}{x}$$

on the interval $[1/2, 3]$.

**Solution**: First we find the critical points: $f'(x) = 1 - 1/x^2$ so the critical points are $x = 1$ and $x = -1$. The latter value, $x = -1$, is not in the interval, so we consider $x = 1$ and the endpoints $x = 1/2$, $x = 3$. Now we compare the values of $f$ at each of these points: $f(1/2) = 5/2$, $f(1) = 0$, and $f(3) = 10/3$, so $f(1)$ is the absolute minimum and $f(3)$ is the absolute maximum.

The absolute maximum can occur at the endpoints or at a critical point – it all depends on the function in question.

## 3. Optimization on the Natural Domain

If the natural domain is not a closed interval, once needs to consider limits of various values. For instance, $f(x) = 1/x$ has no absolute extrema because the limits to the left and right of zero are $-\infty$ and $\infty$, so there can be no absolute maximum. Taking limits is analogous to examining the values of the endpoints and they must be checked to make sure that you do not incorrectly conclude that you have an absolute extrema.