Version A

Lecture 16 Solutions

Solutions

1.

A demographer studying a certain community uses an exponential model $P(t) = P_0 e^{kt}$ for the population, where *t* is the number of years after 1990. If the population is 100000 in 1990 and 158000 in 2001, at what annual percentage rate is the population increasing? Round your final answer to two decimal places.

The population is increasing at the annual rate of ____ %.

Solution:

For simplicity, measure the population $P(t)$ in thousands of individuals. Since the population is 100000 when $t = 0$ (1990) and 158000 when $t = 11$ (2001), we have

$$P_0 = P(0) = 100$$

and

$$P(11) = 100e^{k(11)} = 158$$

$$e^{k(11)} = \frac{158}{100} = 1.58$$

$$\ln\left(e^{k(11)}\right) = \ln(1.58) \qquad \textit{take logarithms on both sides}$$

$$k(11) = 0.4574 \qquad \textit{use } \ln(e^x) = x$$

$$k = \frac{0.4574}{11} = 0.0416$$

Thus, the population is increasing at the annual rate of $100k = 4.16\%$.

Correct answer is: 4.16

2.

Biologists have determined that when sufficient space and nutrients are available, the number of bacteria in a culture grows exponentially. Suppose that 1000 bacteria are initially present in a certain culture and that 3000 are present 20 minutes later. How many bacteria will be present at the end of 2 hours?

____ bacteria will be present at the end of 2 hours.

Solution:

Let $Q(t)$ denote the number of bacteria present after t minutes. Since the number of bacteria grows exponentially and since 1000 bacteria were initially present, you know that Q is a function of the form

$$Q(t) = 1000e^{kt}$$

Since 3000 bacteria are present after 20 minutes, it follows that

$$3000 = 1000e^{20k} \quad \text{or} \quad e^{20k} = 3$$

To find the number of bacteria present at the end of 2 hours (120 minutes), compute $Q(120)$ using the power law for exponents as follows:

$$Q(120) = 1000e^{120k} = 1000(e^{20k})^6 = 1000(3)^6 = 729000$$

That is, 729000 bacteria will be present at the end of 2 hours.

Correct answer is: 729000

3.

The amount of a sample of a radioactive substance remaining after t years is given by a function of the form $Q(t) = Q_0e^{-0.0001t}$. At the end of 5,000 years, 81 grams of the substance remain. How many grams were present initially? Round your answer to the nearest hundredth gram.

Your Answer: ____ grams

Solution:

$$Q(t) = Q_0e^{-0.0001t}$$

When $t = 0$, $Q = Q_0$ grams and when $t = 5000$ years $Q = 81$ grams.

$81 = Q_0 e^{(-0.0001)(5000)} = Q_0 e^{-.5}$ or $Q_0 = 81 e^{.5} = 133.55$ grams.

Correct answer is: 133.55

4.

A radioactive substance decays exponentially. If 600 grams of the substance were present initially and 400 grams are present 50 years later, how many grams will be present after 150 years? Round your answer to the nearest tenth of a gram if needed.
Your Answer: ____ grams

Solution:

Since the decay is exponential and 600 grams were present initially,

$$Q(t) = 600e^{-kt}$$

Also $Q(50) = 600e^{-50k} = 400$, so $e^{-50k} = \frac{2}{3}$

$$\begin{aligned} \text{Now, } Q(150) &= 600e^{-150k} \\ &= 600\left(e^{-50k}\right)^3 \\ &= 600\left(\frac{2}{3}\right)^3 \\ &= 177.8 \text{ grams} \end{aligned}$$

Correct answer is: 177.8

5.

The rate at which a postal clerk can sort mail is a function of the clerk's experience. Suppose the postmaster of a large city estimates that after *t*

months on the job, the average clerk can sort $Q(t) = 820 - 440e^{-0.7t}$ letters per hour. How many letters can a new employee sort per hour?

The number of letters a new employee can sort per hour is ____ .

Solution:

The number of letters a new employee can sort per hour is

$$Q(0) = 820 - 440e^{0} = 380$$

Correct answer is: 380

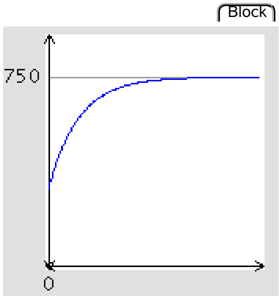
6.

The rate at which a postal clerk can sort mail is a function of the clerk's experience. Suppose the postmaster of a large city estimates that after t months on the job, the average clerk can sort $Q(t) = 750 - 440e^{-0.7t}$ letters per hour. Approximately how many letters will the average clerk ultimately be able to sort per hour?

The average clerk will ultimately be able to sort approximately ____ letters per hour.

Solution:

As t increases without bound, Q(t) approaches 750. Hence, the average clerk will ultimately be able to sort approximately 750 letters per hour. The graph of the function $Q(t)$ is sketched in figure blow.



Worker efficiency $Q(t) = 750 - 440e^{-0.7t}$.

Correct answer is: 750

7.

Public health records indicate that *t* weeks after the outbreak of a certain form influenza, approximately $Q(t) = \frac{100}{1 + 19e^{-1.9t}}$ thousand people had caught the c How many had it 7 weeks later? Round your intermediate calculations to three decimal places and your final answer to the nearest whole number.

About ____ had contracted the disease by the seventh week.

Solution:

When $t = 7$,

$$Q(7) = \frac{100}{1 + 19e^{-1.9(7)}} \approx 99.997$$

so about 99997 had contracted the disease by the seventh week.

Correct answer is: 99997

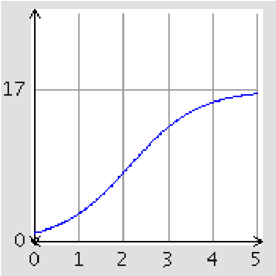
8.

Public health records indicate that *t* weeks after the outbreak of a certain form influenza, approximately $Q(t) = \frac{17}{1 + 16e^{-1.3t}}$ thousand people had caught the c If the trend continues, approximately how many people will eventually contrac disease?

Approximately ____ people will eventually contract the disease.

Solution:

Since $Q(t)$ approaches 17 as t increases without bound, it follows that approximately 17000 people will eventually contract the disease. For reference, the graph is sketched in the figure below.



The spread of an epidemic $Q(t) = \dfrac{17}{1 + 16e^{-1.3t}}$.

Correct answer is: 17000

9.

A drink is taken outside on a cold winter day when the air temperature is -3°C. According to a principle of physics called Newton's law of cooling, the temperature *T* (in degrees Celsius) of the drink *t* minutes after being taken outside is given by a function of the form

$$T(t) = -3 + Ae^{-kt}$$

where *A* and *k* are constants. Suppose the temperature of the drink is 48°C when it is taken outside and 20 minutes later, is 15°C. Use this information to determine *A* and *k*.

A = _____ , *k* = _____

Solution:

Since the temperature is 48°C when $t = 0$, we have

$$T_0 = T(0) = -3 + Ae^{-k(0)} = 48$$

$$-3 + A = 48$$

$$A = 51$$

And since the temperature is 15°C when $t = 20$, we have

$$T(20) = -3 + 51e^{-k(20)} = 15$$

$$51e^{-k(20)} = 15 + 3 = 18$$

$$e^{-k(20)} = \frac{18}{51} = \frac{6}{17}$$

$$\ln\left(e^{-k(20)}\right) = \ln\left(\frac{6}{17}\right) \quad \textit{take logarithms on both sides}$$

$$-k(20) = \ln\left(\frac{6}{17}\right) \quad \textit{use } \ln(e^x) = x$$

$$k = -\frac{1}{20}\ln\frac{6}{17} = \frac{1}{20}\ln\frac{17}{6}$$

Correct answer is: $A = 85,\ k = \frac{1}{20}\ln\frac{17}{6}$

10.

The ratio of ^{14}C to ^{12}C in a sample (e.g., a fossil or an artifact) is given by a function of the form $R(t) = R_0e^{-kt}$. The half-life of ^{14}C is 5730 years. By comparing $R(t)$ to R_0, archaeologists can estimate the age of the sample.

An archaeologist has found a fossil in which the ratio of ^{14}C to ^{12}C is $\frac{8}{79}$ the ratio found in the atmosphere. Approximately how old is the fossil? Round your intermediate results to six decimal places and your final answer to the nearest hundred.

The fossil is approximately ____ years old.

Solution:

The age of the fossil is the value of t for which $R(t) = \frac{8}{79}R_0$; that is, for which

$$\frac{8}{79}R_0 = R_0e^{-kt}$$

Dividing by R_0 and taking logarithms, you find that

$$\frac{8}{79} = e^{-kt}$$

$$\ln \frac{8}{79} = -kt$$

and

$$t = \frac{-\ln \frac{8}{79}}{k} = \frac{\ln 79 - \ln 8}{k}$$

The half-life h satisfies $h = \frac{\ln 2}{k}$, and since ^{14}C has half-life $h = 5730$ years, you have

$$k = \frac{\ln 2}{h} = \frac{\ln 2}{5730} \approx 0.000121$$

Therefore, the age of the fossil is

$$t = \frac{\ln 79 - \ln 8}{k} = \frac{\ln 79 - \ln 8}{0.000121} \approx 18900$$

That is, the fossil is approximately 18900 years old.

Correct answer is: 18900

11.

If $1100 is invested at 8% annual interest, compounded continuously, how long will it take for the investment to double? Round your final answer to two decimal places.

Your Answer: ____ years

Solution:

With a principal of $1100, the balance after t years is $B(t) = 1100e^{0.08t}$, so the investment doubles when $B(t) = \$2200$; that is, when

$$2200 = 1100e^{0.08t}$$

Dividing by 1100 and taking the natural logarithm on each side of the equation, we get

$$2 = e^{0.08t}$$

$\ln 2 = 0.08t$

$$t = \frac{\ln 2}{0.08} \approx 8.66 \text{ years}$$

Correct answer is: 8.66