# MATH 234 BL1 LECTURE 17 NOTES

## 5.2 INTEGRATION

We discussed differential equations last time and looked at solutions for several models pertaining to population biology, supply and demand, learning, and others. To find solutions to differential equations in general, we need to be able to reverse the process of differentiation, called **anti-differentiation** or **integration**.

Integration has many other applications as well, the most central being that it provides a method of continuous summation, a way to aggregate the collective effect of a continuous function or process, such as the area beneath a curve, the cummulative work applied by a spring, and many other quantities such as consumer and producer surpluses in economics. This is the second fundamental problem of calculus – the area problem – and it will turn out to be inversely related to the first fundamental problem (that of rates and tangent lines).

## 1. Area Under a Curve

Before we jump into any theory, let's discuss the idea behind finding the area under a curve and see how it is the reverse of the process of differentiation. Suppose we have a function $f(x)$ on a closed interval $[a, b]$ and we want a function $A(x)$ that gives us the area under the function over the interval $[a, x]$. As this is calculus class, let's look at the derivative of the function $A$. Suppose we step a small distance $h$ away from the value $x$ to a new point $x + h$. For small $h$, this adds approximately a rectangle's worth of area, of width $h$ and height $f(x)$ to the the area under the function $f$ from $a$ to $x$. Then we have that:

$$\begin{aligned} A'(x) &= \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \\ &\approx \lim_{h\to 0} \frac{f(x)h}{h} \\ &= f(x) \end{aligned}$$

In other words, the derivative of the area function $A$ of $f$ is just the function $f$ itself! To find the area function we need to *reverse* differentiation, or *anti-differentiate* the function $f$.

## 2. Antiderivatives

To differentiate a function $f$ we have to find a function $F$, called the antiderivative, whose derivative is $f$; in equations, we need $F'(x) = f(x)$.

Differentiation formulas are also antidifferentiation formulas, as we will see shortly.

**Example 1.** Find a function whose derivative is $2x$.

**Solution**: We know that by the power rule, the derivative of $F(x) = x^2$ is $f(x) = 2x$, so $F(x) = x^2$ is the desired antiderivative. We have a special notation for antiderivatives called the **integral**. In this case, we would say that

$$\int 2x\,dx = x^2$$

which means that $x^2$ is *an* antiderivative of $2x$, and this is because

$$\frac{d}{dx}[x^2] = 2x.$$

It is also the case that

$$\frac{d}{dx}[x^2+1] = 2x,$$

so we see that *antiderivatives are not unique*, but we can say the following:

Any two antiderivatives of a function $f$ differ only by a constant.

Hence, it is more proper to say that *the* antiderivative of $2x$ is $x^2+C$ where $C$ is a constant depending on the context of the problem called the **constant of integration**. Notationally, we would say that

$$\int 2x\,dx = x^2 + C.$$

**Remark 1.** *The derivative of a constant $C$ is zero, so the antiderivative of zero is a constant, btu we don't know which one! We can understand this as follows: another way to think of the constant of integration is a gain of information. Since the derivative is a linearization process (giving the tangent line) and the derivative of a constant is zero, it loses information. Completely reversing this process requires a gain of information (the constant of integration) but that information is contextually dependent, because we cannot know which constant we lost in general. Since we can always add zero to something without changing it, and the antiderivative of a zero is a constant, we need the constant of integration.*

How can we find the antiderivatives in general? By reversing differentiation rules. For example, we have by the power rule

$$\frac{d}{dx}[x^{n+1}] = (n+1)x^n,$$

and moving the constant $(n+1)$ to the left and inside the differentiation operator (as long as $n \neq -1$ to avoid dividing by zero!) we have

$$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] = x^n,$$

Viewing this as an antidifferentiation formula, we would say that

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad \text{for } n \neq -1,$$

which means that $\frac{x^{n+1}}{n+1} + C$ is the antiderivative of $x^n$. What about when $n = -1$? This is the case of $f(x) = \frac{1}{x}$. What function has derivative $\frac{1}{x}$? The natural log! This means that

$$\int \frac{1}{x} \, dx = \ln x + C.$$

**Example 2.** Some examples:

(1) $\int x^3 \, dx = \frac{x^4}{4} + C$

(2) $\int x^{99} \, dx = \frac{x^{100}}{100} + C$

(3) $\int \frac{1}{x^2} \, dx = \int x^{-2} \, dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

(4) $\int 4 \, dx = 4x + C$

## 3. Antidifferentiation Rules

Antiderivatives work much like derivatives do. We can pull out constants

$$\int cf(x) \, dx = c \int f(x) \, dx,$$

and split over sums

$$\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx,$$

but not over products

$$\int f(x) * g(x) \, dx \neq \int f(x) \, dx * \int g(x) \, dx \quad \text{in general.}$$

## 4. Finding an Area

Let's find the area underneath the function $f(x) = x^2$ on the interval $[0, 2]$. First we need to find the area function $A$ by antidifferentiation:

$$A(x) = \int x^2 \, dx = \frac{1}{3}x^3 + C.$$

Although we do not know the constant $C$, we do not need it in this case, because the area under the curve from $x = 0$ to $x = 2$ is given by

$$A(2) - A(0) = [\frac{1}{3}(2)^3 + C] - [\frac{1}{3}(0)^3 + C] = \frac{8}{3} + C - C = \frac{8}{3}.$$

Notice that the constant $C$ was canceled by the subtraction. This is so common that we have a special notation: If $F'(x) = f(x)$ then we write

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

and also

$$\int_a^b f(x)\, dx = F(x)|_a^b.$$

This formula is known as **the fundamental theorem of calculus**. It connects the two major techniques of calculus (differentiation and integration) and the two fundamental problems (finding tangent lines and areas).

We can also obtain more general formulas. For instance, we know that the area of a triangle of height $h$ and base $b$ is $A = \frac{1}{2}bh$. We can see this by considering the area of the function $f(x) = \frac{h}{b}x$ (the line defining the hypotenuse) over the interval $[0, b]$. By integrating, we find that

$$A = \int_0^b \frac{h}{b}x\, dx = \frac{h}{b}\int_0^b x\, dx = \frac{h}{b}\frac{1}{2}x^2|_0^b,$$

and after plugging in the endpoints we see that

$$A = \frac{h}{b}\frac{1}{2}x^2|_0^b = \frac{h}{b}\frac{1}{2}(b)^2 - \frac{h}{b}\frac{1}{2}(0)^2,$$

and so

$$A = \frac{h}{b}\frac{1}{2}b^2 = \frac{1}{2}bh.$$