MATH 234 BL1 LECTURE 19 NOTES

5.2, 5.3 MORE SUBSTITUTION, INITIAL VALUE PROBLEMS

1. More Examples of Substitution

Example 1. Find the antiderivative

$$\int \sqrt{3x+5}\,dx.$$

Solution: It would be nice if $3x+5$ was just an x so we could use the power rule, so let's try $u = 3x+5$; then $du = 3dx$ and we have that

$$\int \sqrt{3x+5}\,dx = \frac{1}{3}\int \sqrt{u}\,du = \frac{1}{3}\frac{2}{3}u^{3/2} + C = \frac{2}{9}(3x+5)^{3/2} + C.$$

Example 2. Find the antiderivative

$$\int x^2 e^{x^3}\,dx.$$

Solution: In this case, we recognize that the derivative of x^3 is lurking out front. Also notice that x^3 is the inner function in the composition e^{x^3}. Let $u = x^3$; then $\frac{du}{dx} = 3x^2$, so the integral becomes

$$\int x^2 e^{x^3}\,dx = \int x^2 e^u \frac{du}{3x^3} = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$$

We'll tackle the following in class together:

$$\int \frac{1}{1+e^{-x}}\,dx$$

2. Substitution with Definite Integration

Care must be taken with definite integrals. Recall the fundamental theorem of calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

When we substitute to a new variable u, we need to change the endpoints as well. Suppose we have

$$\int_a^b f'(g(x))g'(x)\,dx,$$

and we let $u = g(x)$. Then we have that

$$\int_{u(a)}^{u(b)} f'(u)\,du = F(u(b)) - F(u(a)) \neq F(b) - F(a).$$

Example 3. Find the antiderivative

$$\int_1^e \frac{\ln x}{x}\,dx.$$

Solution: In this case, it would be nice if the $\ln x$ would go away, so let's try $u = \ln x$. Then $\frac{du}{dx} = \frac{1}{x}$, so we subsitute $du = xdx$ into the integral, but this time we need to change the endpoints: $u(1) = \ln(1) = 0$, $u(e) = \ln(e) = 1$, and so the integral becomes

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 \frac{u}{x}\,xdu = \int_0^1 u\,du = \frac{1}{2}u^2|_0^1 = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}.$$

Example 4. Find the antiderivative

$$\int_3^8 \sqrt{x+1}\,dx.$$

Solution: Let $u = x + 1$ which gives $du = dx$. Changing the endpoints, we have that $u(3) = 3+1 = 4$ and $u(8) = 8+1 = 9$, so the integral becomes

$$\int_3^8 \sqrt{x+1}\,dx = \int_4^9 \sqrt{u}\,du = \frac{2}{3}u^{3/2}|_4^9 = \frac{2}{3}(9^{3/2} - 4^{3/2}) = \frac{2}{3}(27 - 8) = \frac{38}{3}.$$

3. Initial Value Problems

Recall that an antiderivative of a function f is a function F with the property that $F'(x) = f(x)$. We write the *indefinite integral*

$$\int f(x)\,dx = F(x) + C,$$

where C is the constant of integration, to be determined contextually (if needed). We can write the above equation as a differential equation,

$$\frac{dF}{dx} = f(x),$$

and if we supply some initial information so that we can determine C, such as $F(0) = 2$, we call it an *initial value problem.*

Example 5. If the velocity of a car is given by $v(t) = 22t - t^2$ (t in seconds), what is the position function? If the car was initially at $x = 2$, where has it traveled 5 seconds later?

Solution: The initial value problem is $\frac{dx}{dt} = v(t) = 22t - 3t^2$ with the initial value $x(0) = 2$. The solution is an antiderivative

$$x(t) = \int v(t)\,dt = \int 22t - 3t^2\,dx = 11t^2 - t^3 + C.$$

We determine C by using the value $x(0) = 2$: $2 = x(0) = 11(0)^2 - (0)^3 + C$, so $C = 2$. The car travels to the point $x(5) = 11(5)^2 - (5)^3 + 2 = 152$.

We can also solve more general differential equations of a certain form. A differential equation is called separable if it can be written in the form

$$\frac{dy}{dx} = \frac{f(y)}{g(x)}.$$

The solution is given by integrating and solving algebraically for y. Let's consider the example of exponential growth which we discussed a few lectures back. It has differential equation

$$\frac{dQ}{dt} = kQ.$$

To solve, we integrate

$$\int \frac{1}{Q}\,dQ = \int k\,dt,$$

which gives $\ln Q = kt + C$. Solving for Q gives $Q = e^{kt+C} = e^C e^{kt} = Q_0 e^{kt}$, the solution given before. All the differential equations we discussed previously are of this form and have similar solutions, including the learning and logistics model.