MATH 234 BL1 LECTURE 20 NOTES

5.5 SOME APPLICATIONS OF INTEGRATION

1. Area Between Curves

We can calculate the area between curves as well as just the area under a curve. The area between two curves is the area under the first curve minus the area under the second curve. In equations,

$$\text{Area between } f(x) \text{ and } g(x) \text{ over } [a,b] = \int_a^b f(x) - g(x)\,dx,$$

assuming that $f(x) \geq g(x)$. If the curves intersect, we can find the points a and b by setting them equal; otherwise they must be given.

Example 1. Find the area between $f(x) = x$ and $g(x) = x^2$.

Solution: First let's find the points of intersection by setting $x = f(x) = g(x) = x^2$. Factoring, we obtain $0 = x^2 - x = x(x-1)$, so the intersection points are 0 and 1. To find the area, we note that over the interval $[0,1]$ that $x \geq x^2$, so the integral we need is

$$\int_0^1 x - x^2\,dx = \frac{1}{6}.$$

2. Average Value

We can use integration to calculate the average value of a continuously varying quantity. Recall that the average $\bar{x}$ of n values $x_1, x_2, \ldots x_n$ is

$$\bar{x} = \frac{x_1 + x_2 + \cdots x_n}{n}.$$

The analogous formula for a function f on a closed interval $[a,b]$ is

$$\text{average value of } f = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 2. Find the average value of $f(x) = 1 - x^2$ on the interval $[-1,1]$.

Solution: The integral is

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{1-(-1)}\int_{-1}^1 1 - x^2\,dx = \frac{1}{2}(x - \frac{x^3}{3})|_{-1}^1 = \frac{2}{3}$$

3. Volume of Solid of Revolution

We can also use calculus to compute volume of solids formed by revolving a curve $y = f(x)$ around the x-axis using the following formula:

$$V = \int_a^b \pi [f(x)]^2 \, dx$$

Example 3 (Volume of a cylinder)**.** If we rotate the constant function $y = r$ about the x-axis over the interval $[0, h]$, it traces out a cylinder of radius r and height h (laying on its side). Let's find the volume.

Solution: The integral is

$$V = \int_a^b \pi [f(x)]^2 \, dx = \int_0^h \pi r^2 \, dx = \pi r^2 \int_0^h dx = \pi r^2 h.$$

Example 4 (Volume of a cone)**.** Let's try the same problem for a cone of radius r and height h. This time we are working with the curve $y = \frac{r}{h} x$ over the interval $[0, h]$ as we did with the area of a triangle.

Solution: The integral is

$$V = \int_a^b \pi [f(x)]^2 \, dx = \int_0^h \pi \frac{r^2}{h^2} x^2 dx = \pi \frac{r^2}{h^3} \frac{h^3}{3} = \frac{1}{3} \pi r^2 h.$$