Version A

Lecture 21 Solutions

Solutions

1.

Find the Gini index for the Lorentz curve $L(x) = 0.27x^2 + 0.73x$.
Gini index = ____

Solution:

The Gini index is given by the formula

$$\text{Gini index} = 2\int_0^1 [x - L(x)]\,dx$$

$$= 2\int_0^1 [x - (0.27x^2 + 0.73x)]\,dx$$

$$= 2\left[-0.27\left(\frac{x^3}{3}\right) + 0.27\left(\frac{x^2}{2}\right)\right]\Bigg|_0^1$$

$$= 0.09$$

Correct answer is: 0.09

2.

$p = D(q)$ is the price (dollars per unit) at which q units of a particular commodity will be demanded by the market (that is, all q units will be sold at this price), and q_0 is a specified level of production. Find the price $p_0 = D(q_0)$ at which q_0 units will be demanded so that you can compute the corresponding consumers' surplus CS.

$$D(q) = 47e^{-0.01q};\ q_0 = 9 \text{ units}$$

Round your intermediate calculations and final answer to two decimal places.

The consumers' surplus is $ ____ .

Solution:

$$\begin{aligned} p_0 &= D(q_0) \\ &= 47e^{-0.01(9)} \\ &\approx 42.95 \text{ dollars per unit} \end{aligned}$$

Using $p = 42.95$ and $q_0 = 9$, we find that the consumer's surplus is

$$\begin{aligned} CS &= \int_0^9 (47e^{-0.01q})\, dq - (42.95)(9) \\ &= 47\left(\frac{e^{-0.01q}}{-0.01}\right)\Bigg|_0^9 - (42.95)(9) \\ &= -4700[e^{-0.01(9)} - e^0] - (42.95)(9) \\ &\approx 404.52 - 386.55 = 17.97 \end{aligned}$$

Thus, the consumers' surplus is $17.97.

Correct answer is: 17.97

3.

$p = S(q)$ is the price (dollars per unit) at which q units of a particular commodity will be supplied to the market by producers, and q_0 is a specified level of production. Find the price $p_0 = S(q_0)$ at which q_0 units will be supplied and compute the corresponding producers' surplus *PS*.

$S(q) = 0.7q + 21;\ q_0 = 7$ units

The price $p_0 = S(q_0)$ at which q_0 units will be supplied is $p_0 =$ $ ____ .

The corresponding producers' surplus is *PS* = $ ____ .

Solution:

The producer's supply function is $S(q) = 0.7q + 21$ dollars per unit.

$p_0 = S(7) = 4.9 + 21 = 25.90$

The producer's surplus for $q_0 = 7$ is

$$PS = (7)(25.90) - \int_0^7 0.7q + 21 \, dq = \$17.15$$

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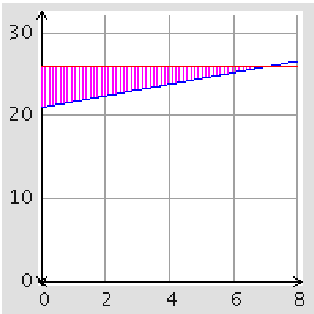


Correct answer is:

***p* = *S*(*q*) is the price (dollars per unit) at which *q* units of a particular commodity will be supplied to the market by producers, and q_0 is a specified level of production. Find the price $p_0 = S(q_0)$ at which q_0 units will be supplied and compute the corresponding producers' surplus *PS*.**

$S(q) = 0.7q + 21$; $q_0 = 7$ units

The price $p_0 = S(q_0)$ at which q_0 units will be supplied is p_0 = $25.90.

The corresponding producers' surplus is *PS* = $17.15.

4.

A manufacturer estimates that q (thousand) products will be purchased (demanded) by wholesalers when the price is

$$p = D(q) = -0.2q^2 + 140$$

dollars per unit, and the same number of products will be supplied when the price is

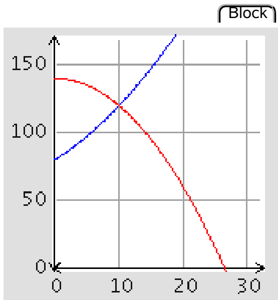
$$p = S(q) = 0.1q^2 + 3q + 80$$

dollars per unit. Determine the producers' surplus at the equilibrium price. Round your intermediate calculations to two decimal places.

The producers' surplus is $ ____ .

Solution:

The supply and demand curves are shown below.



Supply equals demand when

$$\begin{aligned} -0.2q^2 + 140 &= 0.1q^2 + 3q + 80 \\ 0.3q^2 + 3q - 60 &= 0 \\ q &= 10 \quad \text{(reject } q \approx -20\text{)} \end{aligned}$$

and $p = -0.2(10)^2 + 140 = 120$ dollars per unit. Thus, equilibrium occurs at a price of \$120 per unit, and then 10000 products are supplied and demanded.

The producers' surplus is

$$\begin{aligned} \text{PS} &= (120)(10) - \int_0^{10} (0.1q^2 + 3q + 80)\, dq \\ &= (120)(10) - \left[0.1\left(\frac{q^3}{3}\right) + 3\left(\frac{q^2}{2}\right) + 80q\right]\Bigg|_0^{10} \\ &\approx 1200 - 983.33 = 216.67 \end{aligned}$$

or $216670.

Correct answer is: 216670

5.

A manufacturer estimates that q (thousand) products will be purchased (demanded) by wholesalers when the price is

$$p = D(q) = -0.1q^2 + 370$$

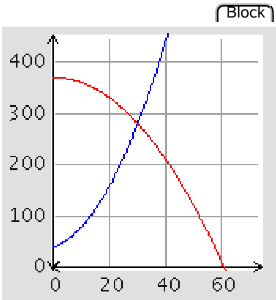
dollars per unit, and the same number of products will be supplied when the price is

$$p = S(q) = 0.2q^2 + 2q + 40$$

dollars per unit. Determine the consumers' surplus at the equilibrium price.
The consumers' surplus is $ ____ .

Solution:

The supply and demand curves are shown to the right.



Supply equals demand when

$$-0.1q^2 + 370 = 0.2q^2 + 2q + 40$$
$$0.3q^2 + 2q - 330 = 0$$
$$q = 30 \quad \text{(reject } q \approx -36.67\text{)}$$

and $p = -0.1(30)^2 + 370 = 280$ dollars per unit. Thus, equilibrium occurs at a price of $280 per unit, and then 30000 products are supplied and demanded.

Using $p_0 = 280$ and $q_0 = 30$, we find that the consumers' surplus is

$$\text{CS} = \int_0^{30} (-0.1q^2 + 370)\,dq - (280)(30)$$

$$= \left[-0.1\left(\frac{q^3}{3}\right) + 370q\right]\Bigg|_0^{30} - (280)(30)$$

$$= 10200 - 8400 = 1800$$

or \$1800000 (since $q_0 = 30$ is really 30000).

Correct answer is: 1800000

6.

Money is transferred continuously into an account at the constant rate of \$1100 per year. The account earns interest at the annual rate of 8% compounded continuously. How much will be in the account at the end of 5 years? Round your final answer to the nearest cent.

The future value of the income stream is \$ ____ .

Solution:

Note that P dollars invested at 8% compounded continuously will be worth $Pe^{0.08t}$ dollars t years later.

To approximate the future value of the income stream, divide the 5−year time interval $0 \le t \le 5$ into n equal subintervals of length Δt years and let t_k denote the beginning of the kth subinterval. Then, during the kth subinterval (of length Δt years),

$$\text{Money deposited} = (\text{dollars per year})(\text{number of years}) = 1100\Delta t$$

If all of this money were deposited at the beginning of the subinterval (at time t_k), it would remain in the account for $5 - t_k$ years and therefore would grow to $(1100\Delta t)e^{0.08(5 - t_k)}$ dollars. Thus,

$$\begin{array}{c}\text{Future value of}\\ \text{money deposited}\\ \text{during } k\text{th subinterval}\end{array} \approx 1100e^{0.08(5 - t_k)}\Delta t$$

The future value of the entire income stream is the sum of the future values of the money deposited during each of the n subintervals. Hence,

$$\text{Future value of income stream} \approx \sum_{k=1}^{n} 1100e^{0.08(5 - t_k)}\Delta t$$

(Note that this is only an approximation because it is based on the assumption that all $1100\Delta t_k$ dollars are deposited at time t_k rather than continuously throughout the kth subinterval.)
As n increases without bound, the length of each subinterval approaches zero and the approximation approaches the true future value of the income stream. Hence,

$$\begin{aligned}
\text{Future value of income stream} &= \lim_{n\to+\infty} \sum_{k=1}^{n} 1100e^{0.08(5 - t_k)}\Delta t \\
&= \int_0^5 1100e^{0.08(5 - t)}dt \\
&= 1100e^{0.4}\int_0^5 e^{-0.08t}dt \\
&= -\frac{1100}{0.08}\, e^{0.4}(e^{-0.08t})\Big|_0^5 \\
&= -13750e^{0.4}(e^{-0.4} - 1) \\
&= -13750 + 13750e^{0.4} \\
&\approx \$6762.59
\end{aligned}$$

Correct answer is: 6762.59

7.

Jane is trying to decide between two investments. The first costs \$1200 and is expected to generate a continuous income stream at the rate of $f_1(t) = 3300e^{0.01t}$ dollars per year. The second investment costs \$4000 and is estimated to generate income at the constant rate of of $f_2(t) = 4400$ dollars per year. If the prevailing annual interest rate remains fixed at 5% compounded continuously over the next 9 years, which investment will generate more net income over this time period? Round your intermediate calculations to two decimal places.

A. The first investment is better.

B. The second investment is better.

Solution:

The net income generated by each investment over the 9–year time period is the present value of the investment less its initial cost. For each investment, we have $r = 0.05$ and $T = 9$.

For the first investment:

$$\begin{aligned}
PV - \text{cost} &= \int_0^9 (3300e^{0.01t})e^{-0.05t}\,dt - 1200 \\
&= 3300\int_0^9 e^{0.01t - 0.05t}\,dt - 1200 \\
&= 3300\int_0^9 e^{-0.04t}\,dt - 1200 \\
&= 3300\left(\frac{e^{-0.04t}}{-0.04}\right)\Bigg|_0^9 - 1200 \\
&= -82500[e^{-0.04(9)} - e^{0}] - 1200 \\
&= 23741.70
\end{aligned}$$

For the second investment:

$$\begin{aligned}
PV - \text{cost} &= \int_0^9 (4400)e^{-0.05t}\,dt - 4000 \\
&= 4400\left(\frac{e^{-0.05t}}{-0.05}\right)\Bigg|_0^9 - 4000 \\
&= -88000[e^{-0.05(9)} - e^{0}] - 4000 \\
&= 27888.72
\end{aligned}$$

Thus, the net income generated by the first investment is \$23741.70, while the second investment generates net income of \$27888.72. The second investment is better.

Correct answer is: B