

MATH 234 BL1 LECTURE 25 NOTES

OPTIMIZING FUNCTIONS OF TWO VARIABLES

1. GRAPHS OF TWO VARIABLE FUNCTIONS

Two variable functions produce a three-dimensional graph, using coordinates x , y , and z and $z = f(x, y)$. For example, $z = \sqrt{x^2 + y^2}$ is a cone formed at the origin. We can also have implicit functions of three variables, such as the sphere $x^2 + y^2 + z^2 = r^2$. One technique to help visualize such functions is with *level curves*, which are the curves in the xy -plane when z is set to a constant. For the double-cone $z^2 = x^2 + y^2$, the level curves are concentric circles.

2. SLOPES OF LEVEL CURVES

The slope of a level curve can be found with partial derivatives. The level curves $z = f(x, y) = C$ has slope

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

To see this, consider the multivariable chain rule, assuming that $x = x(t)$ and $y = y(t)$ are functions of t :

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

Letting $x = t$ and solving for $\frac{dy}{dx}$ gives the formula above.

3. OPTIMIZATION OF FUNCTIONS OF TWO VARIABLES

We are going to proceed as we did for single variable functions, generalizing the second derivative test. The first concept we need is that of a *critical point*. Let's introduce a new notation first:

$$\left. \frac{\partial f}{\partial x} \right|_{(a,b)} = f_x(a, b)$$

$$\left. \frac{\partial f}{\partial y} \right|_{(a,b)} = f_y(a, b)$$

Similarly,

$$\left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(a,b)} = f_{xy}(a, b)$$

$$\frac{\partial^2 f}{\partial y^2}|_{(a,b)} = f_{yy}(a, b)$$

Since we can move in different directions now, we need the slopes of the tangents lines in both coordinate directions to be zero.

Definition 1. A point (a, b) in the domain of $f(x, y)$ is a *critical point* of f if both $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

There are now three possibilities: we could have a maximum, minimum, or *saddle point*. The following test allows us to differentiate between these.

Theorem 1 (Second Partial Derivative Test). *Let $f(x, y)$ be a function whose first and second partial derivatives all exist, and let*

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2;$$

then a critical point (a, b) of f is a:

- (1) *saddle point if $D(a, b) < 0$,*
- (2) *local minimum if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$,*
- (3) *or a local maximum if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$.*

Example 1. Let $f(x, y) = 2x^2 - 3y^2$. The critical point is found by setting $f_x = 4x = 0$ and $f_y = -6y = 0$, so $(0, 0)$ is the only critical point. We have that $D(x, y) = 4 * 6 - 0 * 0 = 24$ and $f_{xx} = 4$, so the critical point is a minimum.

Example 2. Let $f(x, y) = xy$. The critical point is found by setting $f_x = y = 0$ and $f_y = x = 0$, so $(0, 0)$ is the only critical point. We have that $D(x, y) = 0 * 0 - 1 * 1 = -1 < 0$, so the critical point is a saddle point.

Example 3. Let $f(x, y) = xy = \frac{8}{x} + \frac{8}{y}$. The critical point is found by setting $f_x = y - \frac{8}{x^2} = 0$ and $f_y = x - \frac{8}{y^2} = 0$. Rearranging, we have $yx^2 = 8$ and $xy^2 = 8$, so $xy^2 = yx^2$ and hence $xy = (x - y)$. Notice that $x \neq 0$ and $y \neq 0$, so we must have that $x = y$, and so $x = y = 2$, and the only critical point is $(2, 2)$.

We have that $D(x, y) = \frac{-24}{x^3} - \frac{-24}{y^3} - 1 * 1$, and so $D(2, 2) = 8 > 0$. Now $f_{xx} = \frac{-24}{2^3} < 0$, so we have a maximum.

We'll do more examples in class if we have time.