Version A

Lecture 25 Solutions

Solutions

1.

Find all critical points for the function $f(x, y) = 3x^2 + 2y^2$ and classify each as a relative maximum, a relative minimum, or a saddle point.

A. Saddle point at (1, 1)

B. Relative minimum at (1, 1)

C. Relative minimum at (0, 0)

D. Relative maximum at (0, 0)

Solution:

Since

$$f_x = 6x \quad \text{and} \quad f_y = 4y$$

the only critical point of f is (0, 0). To test this point, use the second-order partial derivatives

$f_{xx} = 6 \quad f_{yy} = 4 \quad \text{and} \quad f_{xy} = 0$

to get

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 6(4) - 0^2 = 24$$

That is, D(x, y) = 24 for all points (x, y) and, in particular,

$$D(0, 0) = 24 > 0$$

Hence, f has a relative extremum at (0, 0).

Moreover, since

$$f_{xx}(0, 0) = 6 > 0$$

it follows that the relative extremum at (0, 0) is a relative minimum.

Correct answer is: C

2.

Find all critical points for the function

$$f(x, y) = 3x - x^3 - 3y^2$$

and classify each as a relative maximum and saddle point.

Relative maximum: (____ , ____) and saddle point: (____ , ____).

Solution:

Since

$$f_x = 3 - 3x^2 \quad \text{and} \quad f_y = -6y$$

you find the critical points by solving simultaneously the two equations

$$3 - 3x^2 = 0$$
$$-6y = 0$$

From the second equation, you get $y = 0$ and from the first,

$$3x^2 = 3$$
$$x = 1 \text{ or } -1$$

Thus, there are two critical points (1, 0) and (−1, 0).

To determine the nature of these points, you first compute

$$f_{xx} = -6x \quad f_{yy} = -6 \quad \text{and} \quad f_{xy} = 0$$

and then form the function

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (-6x)(-6) - 0 = 36x$$

Applying the second partials test to the two critical points, you find

$D(1, 0) = 36(1) = 36 > 0$ and $f_{xx}(1, 0) = -6(1) = -6 < 0$

and

$$D(-1, 0) = 36(-1) = -36 < 0$$

so a relative maximum occurs at (1, 0) and a saddle point at (−1, 0). These results are summarized in the table.

Critical point (a, b)	Sign of D(a, b)	Sign of f_{xx}	Behavior at (a,b)
(1, 0) (−1, 0)	+ −	−	Relative maximum Saddle point

Correct answer is: Relative maximum: (1, 0) and saddle point: (− 1, 0).

3.

Find the critical points of the function $f(x, y) = 6 - 8x^2 - 4y^2$ and classify each as a relative maximum, a relative minimum, or a saddle point.

A. f has a relative minimum at (0, 0).

B. f has a relative maximum at (−8, −16).

C. f has a relative maximum at (0, 0).

D. f has a saddle point at (−8, 0).

Solution:

Since

$$f_x = -16x \qquad \text{and} \qquad f_y = -8y$$

the only critical point of f is (0, 0). To test this point, use the second-order partial derivatives

$$f_{xx} = -16 \qquad f_{yy} = -8 \qquad f_{xy} = 0$$

to get

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = (-16)(-8) - 0^2 = 128$$

That is, $D(x, y) = 128$ for all points (x, y) and, in particular,

$$D(0, 0) = 128 > 0$$

Hence, f has a relative extremum at (0, 0). Moreover, since

$$f_{xx}(0, 0) = -16 < 0$$

it follows that the relative extremum at (0, 0) is a relative maximum.

Correct answer is: C

4.

Find the critical points of the given function and classify each as a relativ
a relative minimum, or a saddle point.

$$f(x, y) = 9xy + \frac{512}{x} + \frac{512}{y}$$

The critical point is $\left(\_\_\_\_ , \frac{\_\_\_\_}{\_\_\_\_} \right)$ and

it is a ____ .

Solution:

$$f(x, y) = 9xy + 512x^{-1} + 512y^{-1}$$

$$f_x = 9y - \frac{512}{x^2} = 0 \text{ when } y = \frac{512}{9x^2}.$$

$$f_y = 9x - \frac{512}{y^2} = 0 \text{ when } x = \frac{512}{9y^2}.$$

$$x = \frac{512}{(512/x^2)^2} \text{ is satisfied by } x = 0 \text{ and } x = 8.$$

Discard x = 0 because it is not in the domain of the function.

If $x = 8$, $y = \frac{8}{9}$ also.

$f_{xx} = 1024x^{-3}$,

$f_{yy} = 1024y^{-3}$, and $f_{xy} = 9$

$$D(8, \frac{8}{9}) = \left(\frac{1024}{512}\right)\left(1024(\frac{729}{512})\right) - 9^2 > 0$$

and $f_{xx}(8, \frac{8}{9}) > 0$, so $(8, \frac{8}{9})$ is a relative minimum.

Correct answer is: The critical point is $\left(8, \frac{8}{9}\right)$ and it is a relative minimum.

5.

Find the critical points of the given function and classify each as a relativ
a relative minimum, or a saddle point. Give your critical points in an asce
$f(x, y) = 8x^3 - 5xy + 8y^3$
A critical point is (____ , ____), it is a ____

and the other critical point is $\left(\frac{\_\_\_\_}{\_\_\_\_}, \frac{\_\_\_\_}{\_\_\_\_}\right)$,

it is a ____ .

Solution:

$f_x = 24x^2 - 5y$

So, $f_x = 0$ when $0 = 24x^2 - 5y$, or $y = \frac{24}{5}x^2$.

$f_y = -5x + 24y^2$

So, $f_y = 0$ when $0 = -5x + 24y^2$

$$0 = -5x + 24\left(\frac{24}{5}x^2\right)^2$$

$$= \frac{13824}{25}x^4 - 5x$$

$= 5x\left(\frac{13824}{125}x^3 - 1\right) = 0,$

or $x = 0, \frac{5}{24}$.

When $x = 0$, $f_x = 0$ when $y = 0$.

When $x = \frac{5}{24}$, $f_x = 0$ when $0 = 24\left(\frac{5}{24}\right)^2 - 5y$, or $y = \frac{5}{24}$.

So the critical points are (0, 0) and $\left(\frac{5}{24}, \frac{5}{24}\right)$.

$f_{xx} = 48x$, $f_{yy} = 48y$, $f_{xy} = -5$.

For the point (0, 0),
$D = 48(0)48(0) - (-5)^2 < 0$. So, (0, 0) is a saddle point.

For the point $\left(\frac{5}{24}, \frac{5}{24}\right)$,

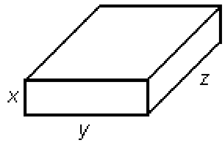
$D = 48\left(\frac{5}{24}\right)48\left(\frac{5}{24}\right) - (-5)^2 > 0$

and $f_{xx} > 0$, so $\left(\frac{5}{24}, \frac{5}{24}\right)$ is a relative minimum.

Correct answer is: A critical point is (0, 0), it is a saddle point and the other critical point is $\left(\frac{5}{24}, \frac{5}{24}\right)$, it is a relative minimum.

6.

Suppose you wish to construct a rectangular box with a volume of 32 ft^3. Three different materials will be used in the construction. The material for the sides costs \$1 per square foot, the material for the bottom costs \$3 per square foot, and the material for the top costs \$5 per square foot. What are the dimensions of the least expensive such box?



x = ____ feet, y = ____ feet, and z = ____ feet

Solution:

Let the box be x inches deep, y inches long, and z inches wide where x, y, and z are all positive, as indicated in the figure above.

Then the volume of the box is V = 32 = xyz or $z = \frac{32}{xy}$

and the total cost is given by

$$C = 1(2xy + 2xz) + 3yz + 5yz = 2xy + 2xz + 8yz$$

Substitute $z = \frac{32}{xy}$ into the cost equation

$$f(x, z) = 2xy + 2x\left(\frac{32}{xy}\right) + 8y\left(\frac{32}{xy}\right) = 2xy + \frac{64}{y} + \frac{256}{x}$$

Compute the partial derivatives

$$f_x = 2y - \frac{256}{x^2} \quad \text{and} \quad f_y = 2x - \frac{64}{y^2}$$

and set them equal to zero to get

$$2y - \frac{256}{x^2} = 0 \quad \text{and} \quad 2x - \frac{64}{y^2} = 0$$

Then solve these equation simultaneously to get

$$x = 8 \quad \text{and} \quad y = 2$$

It follows that (8, 2) is the only critical point of f.

Next apply the second partials test. Since

$$f_{xx} = 512 \quad f_{yy} = 128 \quad \text{and} \quad f_{xy} = 2$$

x^3 y^3

you get

$$D(x, z) = f_{xx}\, f_{yy} - (f_{xy})^2 = \left(\frac{512}{x^3}\right)\left(\frac{128}{y^3}\right) - (2)^2$$

Because you have

$$D(8, 2) = \left(\frac{512}{8^3}\right)\left(\frac{128}{2^3}\right) - (2)^2 = 12 > 0$$

and

$$f_{xx}(8, 2) = \frac{512}{8^3} = 1 > 0$$

it follows that f has a (relative) minimum when x = 8 and y = 2.

Substituting these values into the volume equation xyz = 32, you find that z = 2.

Thus, the minimal cost occurs when the box is 8 feet deep, 2 feet long and 2 feet wide.

Correct answer is: x = 8 feet, y = 2 feet, and z = 2 feet
