MATH 234 BL1 LECTURE 26 NOTES

LAGRANGE MULTIPLIERS

1. Constrained Optimization

The method of Lagrange multipliers allows the optimization of multivariable functions in the presence of one or more constraints. Given a function $f(x, y)$ to optimize constrained b a function $g(x, y) = k$, the relative extrema can be shown to occur at critical points of the functions

$$F(x, y) = f(x, y) - \lambda[g(x, y) - k],$$

where the symbol λ is pronounced 'lambda' and is called the *multiplier.* To find the critical points of F, we compute the partial derivatives

$$F_x = f_x - \lambda g_x = 0$$
$$F_y = f_y - \lambda g_y = 0$$
$$F_\lambda = -(g - k) = 0$$

and so we get a set of equations

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g = k,$$

the solutions of which correspond to the critical points of F.

Example 1. Find the maximum and minimum values of $f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 1$.

Solution: The constraint function $g(x, y) = x^2 + y^2 = 1$, so $k = 1$. First we use the three equations above to determine the critical points.

$$f_x = \lambda g_x : y = \lambda 2x$$
$$f_y = \lambda g_y : x = \lambda 2y$$
$$x^2 + y^2 = 1$$

Notice from the first two equations that if either $x = 0$ or $y = 0$ then so does the other, but then $x^2 + y^2 = 0$, which should be 1, so neither $x = 0$ or $y = 0$. Rearranging the first two equations, $\frac{y}{x} = 2\lambda = \frac{x}{y}$, so by cross-multiplying we see that $x^2 = y^2$ and so $x = y$ or $x = -y$. The constraint equation then gives $1 = x^2+y^2 = 2x^2$, so $x = \pm\frac{1}{\sqrt{2}}$ and we get four solutions: $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Plugging these values into f, we see that the maximum occurs at $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{1}{2}$ and the minimum at $f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -\frac{1}{2}$.

2. Why Lagrange Multipliers Works

This is not a formal proof, but should give you an idea why the method works. To find the maximum value of $f(x, y)$ subject to the constraint $g(x, y) = k$, we need to find the highest level curve of f that intersects $g(x, y) = k$. This means that the *slopes* of the level curve and constraint curve need to be equal, and we saw in the last lecture that the slope $\frac{dy}{dx}$ of a level curve $f(x, y) = k$ is given by

$$\frac{dy}{dx} = -\frac{f_x}{f_y}.$$

Setting the two slopes equal, we have that

$$-\frac{f_x}{f_y} = -\frac{g_x}{g_y},$$

and so

$$\frac{f_x}{g_x} = \frac{f_y}{g_y}.$$

Let this common ratio be given by λ. Then we have

$$\frac{f_x}{g_x} = \lambda \qquad \text{and} \qquad \frac{f_y}{g_y} = \lambda,$$

which gives the equations given at the beginning of these notes.

3. Lagrange Multipliers for Three Variables

For a three variable equation and constraint we just need to add another equation for the new variable. If we want to optimize $f(x, y, z)$ subject to the constrain $g(x, y, z) = k$, we need to solve $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$ and $g = k$.

Example 2. Find the maximum and minimum values of $f(x, y) = xyz$ subject to the constraint $x + 2y + 3z = 24$.

Solution: We need to solve the system of equations

$$\begin{aligned} f_x = \lambda g_x &: yz = \lambda \\ f_y = \lambda g_y &: xz = \lambda * 2 \\ f_z = \lambda g_z &: xy = \lambda * 3 \\ x + 2y + 3z &= 24 \end{aligned}$$

Plugging the first equation into the second gives $xz = 2yz$ and factoring leads to $(x - 2y)z = 0$, so either $z = 0$ or $x = 2y$. Similarly, plugging the first equation into the third gives that $y(x - 3z) = 0$.

Now we proceed in cases. First assume that $z = 0$. Then we have that $0 = xz = 2\lambda$ so $\lambda = 0$ which implies that $xy = 3\lambda = 0$, so either $x = 0$ or $y = 0$. If $x = 0$, then the constraint equation gives $24 = x+2y+3z = 2y$, so the solution is $(0, 12, 0)$. If $y = 0$, then the constraint equation gives $24 = x$, so the solution is $(24, 0, 0)$.

Assume next that $z \neq 0$, so $x = 2y$. If $y = 0$ then $z = 0$ and we have the solution $(0, 0, 8)$ by solving for z in the constraint. Otherwise we must have that $x = 3z$, so we can plug into the constrain equation $24 = x + x + x$ to get that $x = 8$, and so then $y = 4$ and $z = \frac{8}{3}$.

Finally, compare the values of all the found critical points: $f(24, 0, 0) = 0 = f(0, 12, 0) = f(0, 0, 8)$ and $f(8, 4, \frac{8}{3}) = \frac{256}{3}$.

We'll do more examples in class if we have time.