RESEARCH STATEMENT

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I am a discrete mathematician, specializing in graph theory and extremal problems. People have a natural fascination with everyday objects that are extreme with respect to some property like height or speed. In much the same way, extremal objects in mathematics are inherently interesting. Extremal problems have also inspired the development of deep and important techniques, such as the probabilistic method and the regularity method.

Extremal problems have applications beyond discrete mathematics. For example, the problem of register allocation in computer science can be cast as a graph coloring problem. The existence of certain extremal structures have important consequences for algorithmic performance guarantees, such as the time complexity of matrix multiplication. My work on an edge-labeling problem leads to a solution of a problem in computability theory (see "Subtrees with Few Path-labels").

The interaction between extremal problems and broader mathematics is bidirectional; the solutions to many extremal problems use beautiful tools from a variety of areas, including linear algebra and Fourier analysis. In some cases, algebraic structures serve as the foundation for the construction of an extremal object.

Extremal problems come in many different flavors. I particularly enjoy probabilistic questions and techniques, combinatorial games, and algorithmic questions. I also enjoy exploring the relationship between different graph parameters. In the following, I provide a detailed summary of my research.

Degree-Ramsey Theory. Ramsey theory consists of a diverse array of results which have a common theme: large objects must contain smaller parts that are highly structured. Consider the classic context of Ramsey's theorem. A *graph* G is a set of *vertices* $V(G)$ and a set of *edges* $E(G)$, each of which is an unordered pair of vertices. A pair of vertices is *adjacent* if the pair is an edge. The *complete graph* K_n is the graph on n vertices in which each of the $\binom{n}{2}$ unordered pairs is adjacent. The *path* P_n is the graph on n vertices $\{v_1, \ldots, v_n\}$ in which each consecutive pair of vertices $\{v_i, v_{i+1}\}$ is an edge. The *cycle* C_n is similar except that $\{v_n, v_1\}$ is also an edge. An *s-edge-coloring* is a function that assigns to each edge a color from a set of size s. Let H be a "host" graph and let G be a "target" graph. If every s-edge-coloring of H contains a monochromatic copy of G as a subgraph, then we write $H \xrightarrow{s} G$. When $s = 2$, we omit the superscript and simply write $H \to G$. For example, $K_3 \to P_3$ because a 2-edge-coloring of the triangle K_3 uses the same color on two of its three edges.

Ramsey's theorem states that for every graph G and every s, there exists an n such that $K_n \xrightarrow{s} G$. Informally, if the host graph is large enough and dense enough, a monochromatic copy of the target graph is forced in every s-edge-coloring of the host's edges. The *Ramsey number* of a graph G, denoted $R(G)$, is $\min\{|V(H)| : H \to G\}$; similarly, we define $R(G; s)$ to be $\min\{|V(H)| : H \xrightarrow{s} G\}$. The Ramsey number is a classic graph parameter and has been studied extensively. The principal question is, given a graph G, how large does the host graph H need to be before $H \xrightarrow{s} G$?

I am interested in a related question. If we are willing to allow the host graph H to have arbitrarily many vertices, how dense must we make H before $H \xrightarrow{s} G$? The *degree* of a vertex v in a graph H is the number of edges that contain v. The maximum degree of a vertex in H is denoted by $\Delta(H)$. If H has small maximum degree, then H is sparse. The *degree-Ramsey number* of a graph G, denoted $R_\Delta(G)$, is $\min\{\Delta(H) \colon H \to G\}$, and we define $R_\Delta(G; s)$ to be $\min\{\Delta(H) \colon H \xrightarrow{s} G\}$.

The degree-Ramsey number has been established for some graphs. Burr, Erdős, and Lovász [8] showed that $R_\Delta(K_n) = R(K_n) - 1$ for each n. More recently, Kurek and Ruciński [25] proved the stronger statement that if $H \to K_n$, then H contains a subgraph with average degree at least $R(K_n) - 1$. Burr, Erdős, and Lovász [8] also characterized the set of host graphs H such that $H \to K_{1,n}$, where $K_{m,n}$ is the *complete bipartite graph* with partite sets of sizes m and n. Their characterization immediately implies that $R_\Delta(K_{1,n})$ is $2n - 2$ when n is even and $2n - 1$ when n is odd. Kinnersley, West, and I generalized this by obtaining the degree-Ramsey number of the double star $S_{a,b}$, the tree that consists of two adjacent vertices u and v and enough pendant edges so that u has degree a and v has degree b; the graph $K_{1,n}$ arises as $S_{1,n}$.

In studying the size of monochromatic components of 2-edge-colored graphs, Alon et al. [3] gave a short argument that proves $R_\Delta(P_n) \le 4$ for all n. On the other hand, Thomassen [35] proved that every graph with maximum degree at most 3 can be 2-edge-colored so that all monochromatic components are subgraphs of P_6. Consequently, $R_\Delta(P_n) = 4$ for all $n \ge 7$.

The most intriguing problem about the degree Ramsey parameter is whether $R_\Delta(G\,; s)$ is bounded by a function of $\Delta(G)$ and s. Jiang [27] observed that the argument of Alon et al. extends to show that $R_\Delta(T;\, s) \le 2s\Delta(T)$ for each tree T; a *tree* is a connected graph with no cycles. Jiang, West, and I extended this to prove the following.

Theorem. [28] *Let $\mathcal{F}$ be the family of graphs that can be obtained from a tree T by replacing each vertex in T with an independent set and each edge in T with a complete bipartite graph. There exists a function f such that $R_\Delta(G) \le f(\Delta(G))$ for each graph G in $\mathcal{F}$.*

Specializing these techniques to cycles, we proved the following.

Theorem. [28] *If n is even, then $R_\Delta(C_n) \le 108$. For all n, $R_\Delta(C_n) \le 3890$.*

Previously, a result of Haxell et al. [19] implicitly showed that $R_\Delta(C_n)$ is bounded; because their proof uses Szemerédi's regularity lemma, their bound is very large. For very small cycles, Kinnersley, West, and I obtained the value exactly.

Theorem. [20] $R_\Delta(C_3) = R_\Delta(C_4) = 5$.

The difficult part of the above theorem is that $R_\Delta(C_4) > 4$. This follows from the following decomposition theorem.

Theorem. [20] *If H is a graph with $\Delta(H) \le 4$, then H can be decomposed into two graphs with girth at least* 5.

Online Degree-Ramsey Theory. Traditional Ramsey theory can be viewed as a game between two players: Builder and Painter. Builder presents a host graph H to Painter, and then Painter colors the edges of H. Builder wins if Painter's coloring contains a monochromatic copy of some target graph G; otherwise Painter wins. The statement that $R(G) \le k$ is equivalent to the statement that when G is the target graph, Builder has a winning strategy even when Builder is allowed to present only graphs with at most k vertices.

An online variant was introduced by Grytczuk, Hałuszczak, and Kierstead [16], where Builder and Painter take turns. The host graph H starts empty. Builder selects a pair $\{u, v\}$ of non-adjacent vertices (one or both of which may be new), and adds the edge uv to H. Painter responds by coloring uv red or blue. We restrict the graphs that Builder is allowed to present to a monotone family $\mathcal{H}$ of host graphs. Builder must present edges so that at all times, H is a member of $\mathcal{H}$. Builder wins if Painter ever completes a monochromatic copy of the target graph G, and Painter wins otherwise. This defines the *online Ramsey game* $(G, \mathcal{H})$. The fundamental problem of online Ramsey theory is to characterize the games $(G, \mathcal{H})$ in which Builder has a winning strategy.

Each monotone offline Ramsey parameter has an online analogue. The *online degree-Ramsey number*, denoted $R^\circ_\Delta(G)$, is $\min\{k\colon$ builder wins $(G, \mathcal{H}_k)\}$, where $\mathcal{H}_k$ is the family of graphs with maximum degree at most k. Butterfield, Grauman, Kinnersley, Stocker, West, and I [9] studied the online degree-Ramsey number of trees and cycles.

Theorem. [9] *If T is a tree, then $R^{\circ}_{\Delta}(T) \leq 2\Delta(T) - 1$ with equality whenever T contains an adjacent pair of vertices with maximum degree.*

We obtained the following result for cycles.

Theorem. [9] *For each n, $R^{\circ}_{\Delta}(C_n) \in \{4, 5\}$. If n is even, $n = 3$, $337 \leq n \leq 514$, or $n \geq 689$, then $R^{\circ}_{\Delta}(C_n) = 4$.*

We do not know of any cycle with online degree-Ramsey number 5; the first unknown case is C_5. The lower bound $R^{\circ}_{\Delta}(C_n) > 3$ follows from our characterization of the graphs G with online degree-Ramsey number at most 3. It would be interesting to know if the online degree-Ramsey number is bounded by a function of the maximum degree. Of course, if the (offline) degree-Ramsey number is bounded by a function of the maximum degree, then so is the online degree-Ramsey number.

Chromatic Number of Circle Graphs. A *clique* is a set of vertices that are pairwise adjacent, and the *clique number* of a graph G, denoted $\omega(G)$, is the maximum size of a clique in G. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum size of a partition of the vertices into independent sets. Because the vertices of a clique must be placed into distinct parts, it follows that $\chi(G) \geq \omega(G)$ for every graph. While $\chi(G)$ may be arbitrarily large even when $\omega(G) = 2$, there are a number of interesting classes of graphs in which $\chi(G)$ is bounded by a function of $\omega(G)$.

A *circle graph* is a graph G whose vertices are chords of a circle drawn in the plane where two chords u and v are adjacent if they cross. Kostochka and Kratochvíl [22] showed that $\chi(G) \leq 50 \cdot 2^{\omega(G)} - 32\omega(G) - 64$ for each circle graph G, and there are circle graphs with $\omega(G) = k$ and $\chi(G) \geq c \cdot k \log k$ for a constant c. The exponential gap between the two bounds has been open for over a decade. More is known when the clique number is small. If G is a circle graph with $\omega(G) = 2$, then $\chi(G) \leq 5$ [21] and this bound is best possible [1]. Kostochka and I investigated the case that $\omega(G) \leq 3$.

Theorem. [23] *If G is a circle graph with $\omega(G) \leq 3$, then $\chi(G) \leq 40$.*

Prior to our work, the best known bound for the case $\omega(G) \leq 3$ was $\chi(G) \leq 240$, which follows from the result of Kostochka and Kratochvíl. Our proofs make use of a lemma which shows that for a subfamily of circle graphs which do not contain a certain structure, $\chi(G) \leq 2\omega(G) - 1$.

Acyclic Sets in k-majority Tournaments. Let Π be a set of linear orders of a ground set X. The *majority digraph* of Π is the directed graph D on vertex set X with a directed edge $uv \in E(D)$ if more than half of the orders in Π rank u before v. If Π has size k, then D is a *k-majority digraph*. When k is odd, each pair of vertices in D spans a directed edge, and so D is a *k-majority tournament*. A *dominating set* in D is a set S of vertices such that for each vertex $v \in V(D)$, either $v \in S$ or $uv \in E(D)$ for some vertex $u \in S$. The *domination number* of D, denoted $\gamma(D)$, is the smallest size of a dominating set in D. Alon et al. [2] introduced k-majority tournaments and showed that there is a constant c such that $\gamma(D) \leq ck \log k$ for every k-majority tournament D. Moreover, they construct a family of k-majority tournaments $\{D_k\}$ such that $\gamma(D_k) \geq c'k/\log k$ for some constant k.

Let $a(D)$ denote the maximum size of an acyclic set in D, and let $f_k(n)$ be the minimum over all n-vertex k-majority tournaments D, of $a(D)$. Schreiber, West, and I proved the following.

Theorem. [31] *For each n, $f_3(n) \geq \sqrt{n}$. When n is a perfect square, $f_3(n) \leq 2\sqrt{n} - 1$.*

We also show that $f_5(n) \geq n^{1/4}$, and the following for general k.

Theorem. [31] *If $c_k = 3^{-(k-1)/2}$ and $d_k = O(\log \log k / \log k)$, then $n^{c_k} \leq f_k(n) \leq n^{d_k}$.*

Because adding a permutation and its reverse to Π does not change the majority digraph, the construction for 3-majority tournaments yields $f_k(n) \leq 2\sqrt{n} - 1$ whenever $k \geq 3$ and n is a perfect square. This bound is the best known when $k = 5$.

Subtrees with Few Path-labels. One goal in computability theory is to find algorithmic solutions to combinatorial problems, and solutions to problems in computability theory often require combinatorial proofs. Consequently, computability theory and combinatorics enjoy a history of successful collaboration. Recently, I had the opportunity to participate in one such collaboration. The motivating question from computability theory led to a Ramsey-type problem of finding highly structured binary subtrees in large ternary trees; this problem is of independent combinatorial interest.

The following paragraph briefly outlines the motivating question from computability theory and assumes a degree of familiarity with the field. Those who would prefer may safely advance to the following paragraph, where the combinatorial problem and results are discussed. Recall that a function f is *Turing-reducible* to g, denoted $f \leq_T g$, if there is an algorithm that computes f using an oracle for g as a subroutine. A *class* is a set of functions. It is possible to extend the concept of reduction to classes. A class $\mathcal{A}$ is *strongly-reducible* or *Medvedev-reducible* or to a class $\mathcal{B}$, denoted $\mathcal{A} \leq_s \mathcal{B}$, if there is an algorithm Φ such that for each $g \in \mathcal{B}$, equipping Φ with an oracle for g results in an algorithm that computes some function in $\mathcal{A}$. Informally, $\mathcal{A} \leq_s \mathcal{B}$ if there is a uniform way of using oracles for functions in $\mathcal{B}$ to compute a function in $\mathcal{A}$. The class of *diagonally non-recursive functions*, denoted DNR, is the set of all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(n) \neq \varphi_n(n)$ for each $n \in \mathbb{N}$, where φ_n is the partial function computed by the nth Turing machine. For each $k \geq 2$, the class DNR_k is defined to be the subset of DNR consisting of functions whose image is contained in $\{0, \dots, k-1\}$. Simpson [34] asked if every Π^0_1 class of positive measure is strongly-reducible to DNR_3. Downey, Greenberg, Jockusch, and I [13] answered this question in the negative by showing that the class of Kurtz-random (or weakly 1-random) functions is not strongly-reducible to DNR_3. Central to our proof is an analysis of a combinatorial problem on trees.

A rooted tree is *complete* if all leaves are at the same distance from the root, and the *depth* of a complete tree is the common distance between the leaves and the root. A rooted tree is *q-ary* if all non-leaves have q children. If T is a complete tree of depth n in which each edge is labeled with 0 or 1, then reading the edge labels along a path from the root to a leaf yields a *path-label* in $\{0,1\}^n$. Let $L(T)$ be the set of all path-labels that occur along paths in T.

When T is a complete ternary tree of depth n and S is a complete binary subtree of T of depth n, we write $S \sqsubset T$. Given a $\{0,1\}$-edge labeled complete ternary tree T, we seek a binary subtree S with as few path-labels as possible. Let $f(T) = \min\{|L(S)| : S \sqsubset T\}$. Of course, because $L(S) \subseteq \{0,1\}^n$, always $f(T) \leq 2^n$. We define

$$f(n) = \max\{f(T)\colon\ T \text{ is a } \{0,1\}\text{-edge-labeled complete ternary tree of depth } n\}.$$

The computability theory application requires that $\lim_{n\to\infty} f(n)/2^n = 0$. In fact, we the following.

Theorem. [13] *There exist positive constants c_1, c_2, c_3 such that $c_1 2^{\frac{1}{\log_2 3} n} \leq f(n) \leq c_2 2^{n - c_3\sqrt{n}}$ for each n.*

A relatively simple argument shows that $\lim_{n\to\infty} (f(n))^{1/n}$ exists; our bounds on $f(n)$ imply that $1.548 \leq 2^{\frac{1}{\log_2 3}} \leq \lim_{n\to\infty} (f(n))^{1/n} \leq 2$. It would be interesting to improve the bounds on $f(n)$ and shed more light on the value of $\lim_{n\to\infty} (f(n))^{1/n}$.

The techniques in our paper partially extend to natural generalizations. For $2 \leq t \leq p < q$, consider the problem of finding p-ary subtrees of a $\{0, \dots, t-1\}$-edge labeled complete q-ary tree with few path-labels. When T is a complete q-ary tree of depth n and S is a complete p-ary subtree of depth n, we write $S \sqsubset_p T$. Let $f(T;\, p) = \min\{|L(S)| :\ S \sqsubset_p T\}$, and define

$$f(n;\, t, p, q) = \max\{f(T;\, p)\colon\ T \text{ is a } \{0, \dots, t-1\}\text{-edge-labeled complete } q\text{-ary tree of depth } n\}.$$

The behavior of $f(n;\, t, p, q)/t^n$ remains largely unexplored. A simple pigeonhole argument and a straightforward generalization of the techniques for the case $(t, p, q) = (2, 2, 3)$ suffice to show that

$$\lim_{n\to\infty} \frac{f(n;\, t, p, q)}{t^n} = \begin{cases} 0 & \text{if } p < \frac{1}{2}q + 1 \\ 1 & \text{if } p \geq \frac{t-1}{t}q + 1 \end{cases},$$

so the limit is completely determined when $t = 2$. The first unknown case is $(t, p, q) = (3, 3, 4)$. Extending our knowledge of the behavior of $f(n;\, t, p, q)/t^n$ to additional cases promises to provide nice problems for future research.

Computational Complexity of Graph Pebbling. A classic pigeonhole argument shows that every list $a_1, \ldots, a_n$ of n elements of a group Γ of order n contains a non-empty sublist $a_i, \ldots, a_j$ of consecutive elements such that $a_i \cdots a_j = 1_\Gamma$. In 1989, Lemke and Kleitman [26] showed that when $\Gamma = \mathbb{Z}_n$, every list $a_1, \ldots, a_n$ contains a non-empty sublist $a_{i_1}, \ldots, a_{i_k}$ such that $a_{i_1} + \cdots + a_{i_k} = 0$ and $1/|a_{i_1}| + \cdots + 1/|a_{i_k}| \leq 1$, where $|a|$ denotes the order of a in $\mathbb{Z}_n$. Lagarias and Saks proposed the concept of graph pebbling as an alternate, more structural approach to obtaining the result of Lemke and Kleitman. Later in 1989, Chung [11] implemented the proposal of Lagarias and Saks. Since that time, graph pebbling has attracted the attention of numerous combinatorial researchers and become a topic of independent interest.

Graph pebbling studies how distributions of pebbles to the vertices of a graph can change under a sequence of pebbling moves. A *pebbling move* deletes two pebbles from a vertex u and adds a single pebble to a neighbor of u. Given a graph and a distribution of pebbles, a vertex u is *reachable* if some sequence of zero or more pebbling moves terminates with at least one pebble at u. Given a graph, a distribution of pebbles is *solvable* if each vertex is reachable. The *pebbling number* of a graph, denoted $\pi(G)$, is the minimum k such that every distribution of k pebbles to the vertices of G is solvable. The *optimal pebbling number* of a graph, denoted $\pi^\star(G)$, is the minimum number of pebbles in a solvable distribution.

A combinatorial problem's computational complexity is a core property and provides deep insight into the nature of the problem. Clark and I [29] determined the computational complexity of several problems in graph pebbling.

Theorem. [29] *Deciding whether $\pi^\star(G) \leq k$ is NP-complete and deciding whether $\pi(G) \leq k$ is Π_2^{P}-complete.*

The latter result shows that unless the polynomial hierarchy collapses (a remote possibility regarded as only slightly more likely than P=NP), finding the pebbling number of a graph is impossible in polynomial time, even if one is given the power to solve NP-complete problems instantly.

Parity Edge-Colorings of Graphs. A *parity walk* in an edge-colored graph is a walk that traverses each color an even number of times. An edge-coloring is a *strong parity edge-coloring* if every parity walk is closed (starts and ends at the same vertex). A strong parity edge-coloring is *optimal* if it uses the minimum possible number of colors. The *strong parity edge-chromatic number* of a graph G, denoted $\widehat{p}(G)$, is the number of colors in an optimal strong parity edge-coloring.

While Bunde, West, Wu, and I [6] introduced the strong parity edge-chromatic number as a general graph parameter, related concepts were studied earlier. A 1972 result of Havel and Movárek [18] essentially states that $\widehat{p}(P_n) = \lceil \log_2 n \rceil$, where P_n is the path on n vertices. More generally, their results imply that when T is a tree, $\widehat{p}(T)$ is the minimum k such that T is a subgraph of the k-dimensional hypercube.

Complete graphs admit strong parity edge-colorings with nice structure. Let $\mathbb{F}_2^k$ denote the k-dimensional vector space over the field with 2 elements, and let $A \subseteq \mathbb{F}_2^k$. The *canonical edge-coloring* of the complete graph $K(A)$ with vertex set A assigns each edge uv the color $u + v$. The canonical edge-coloring is a strong parity edge-coloring. Indeed, if $u_0, \ldots, u_t$ is a walk in which the jth edge has color c_j, then $u_0 + \sum_{j=1}^{t} c_j = u_0 + \sum_{j=1}^{t}(u_{j-1} + u_j) = u_t$. Also, if $u_0, \ldots, u_t$ is a parity walk, then $\sum_{j=1}^{t} c_j = 0$. Hence, every parity walk is closed.

The canonical edge-coloring of $K(A)$ does not use the 0 vector. Consequently, if n is an integer and m is the smallest power of 2 that is at least n, then $\widehat{p}(K_n) \leq m - 1$, where K_n denotes the complete graph on n vertices. Bunde, West, Wu, and I proved that equality holds.

Theorem. [7] *For each n, $\widehat{p}(K_n) = 2^{\lceil \log_2 n \rceil} - 1$. In fact, every optimal strong parity edge-coloring of K_n is isomorphic to the canonical edge-coloring of $K(A)$ for some $A \subseteq \mathbb{F}_2^k$, where $k = \lceil \log_2 n \rceil$.*

These results strengthen a special case of Yuzvinsky's Theorem from additive combinatorics. For each $r, s \geq 1$, the Hopf–Stiefel function $r \circ s$ is the minimum integer n such that $(x+y)^n$ is in the ideal of $\mathbb{F}_2[x, y]$ generated by x^r and y^s. Equivalently, $r \circ s$ is the minimum integer n such $\binom{n}{k}$ is even for each k with $n - s < k < r$. Yuzvinsky [36] proved that if $A, B \subseteq \mathbb{F}_2^k$, $|A| = r$, $|B| = s$, and $C = \{a + b \,|\, a \in A \text{ and } b \in B\}$, then $|C| \geq r \circ s$, and this is tight. When $A = B$ and both have size r, Yuzvinsky's Theorem states that the set of colors $\{a + b \,|\, a, b \in A \text{ and } a \neq b\}$ used by the canonical edge-coloring of $K(A)$ has size at least $r \circ r - 1$. Because $r \circ r = 2^{\lceil \log_2 r \rceil}$ and the family of strong parity edge-colorings is more general than the family of canonical edge-colorings, our result strengthens this case.

Canonical edge-colorings extend to complete bipartite graphs in a natural way. If $A, B \subseteq \mathbb{F}_2^k$, then the canonical edge-coloring of the complete bipartite graph $K(A, B)$ with partite sets A and B assigns each edge uv the color $u + v$. The set of colors used by the canonical edge-coloring of $K(A, B)$ is $\{a + b \,|\, a \in A \text{ and } b \in B\}$. In its full generality, Yuzvinsky's Theorem states that the canonical edge-coloring of $K(A, B)$ uses $r \circ s$ colors, where $r = |A|$ and $s = |B|$. We conjecture that every strong parity edge-coloring of $K(A, B)$ also needs $r \circ s$ colors, or equivalently that $\widehat{p}(K_{r,s}) = r \circ s$, where $K_{r,s}$ denotes the complete bipartite graph with partite sets of sizes r and s. Proving this conjecture would strengthen all cases of Yuzvinsky's Theorem.

Cycle Spectra of Hamiltonian Graphs. The *cycle spectrum* of a graph G is the set of lengths of cycles in G. A graph on n vertices is *pancyclic* if its cycle spectrum contains all lengths from 3 to n. Let $s(G)$ denote the size of the cycle spectrum of G. In 1960, Ore [32] showed that every n-vertex graph in which every pair of non-adjacent vertices has degree sum at least n is Hamiltonian. Subsequently, Bondy [5] showed that every graph satisfying Ore's condition is either $K_{n/2,n/2}$ or pancyclic. On this basis, Bondy proposed the meta-conjecture that natural sufficient conditions for Hamiltonicity are often also sufficient for pancyclicity.

When G is $(n/2)$-regular, Bondy's result shows that G is pancyclic or $K_{n/2,n/2}$. Conditions that are not strong enough to imply pancyclicity may nevertheless imply a large cycle spectrum. Jacobson and Lehel asked how small the cycle spectrum can be when G is a 3-regular Hamiltonian graph. For n divisible by 6, they constructed a 3-regular Hamiltonian graph G with $s(G) = n/6 + 3$. Jacobson and Lehel, and independently Jiang, proved (but did not publish) the result that $s(G) \geq \sqrt{a(m-n)}$ when G is an n-vertex Hamiltonian graph with m edges, where a is a positive constant. It follows that $s(G) \geq \sqrt{\frac{a}{2}n}$ when G is a 3-regular Hamiltonian graph. Neither Jacobson and Lehel nor Jiang attempted to optimize the constant a.

Using ideas from a paper of Faudree et al. [14], Rautenbach, Regen, West, and I proved the following.

Theorem. [30] *If G is a Hamiltonian graph with n vertices and m edges, then $s(G) \geq \sqrt{\frac{4}{7}(m-n)}$.*

On the other hand, if n is even and $G = K_{n/2,n/2}$, then $s(G) \leq \sqrt{m-n} + 1$. Hence, the largest constant a such that $s(G) \geq \sqrt{a(m-n)} - O(1)$ satisfies $4/7 \leq a \leq 1$. When G is 3-regular, our lower bound yields the following corollary.

Corollary. [30] *If G is a 3-regular Hamiltonian graph, then $s(G) \geq \sqrt{\frac{2}{7}n}$.*

It is believed that a linear lower bound is possible, asymptotically matching the example of Jacobson and Lehel. However, obtaining better bounds in the case that G is 3-regular or even k-regular with $3 < k < n/2$ will require different ideas from those developed in our paper. This question is a direction for future research.

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