Definition The *average value* of a continous function $f(x)$ over the interval $a \leq x \leq b$ is defined as the quantity

$$\frac{1}{b-a}\int_a^b f(x)dx.$$

Definition The *consumers' surplus* for a commodity having demand curve $p = f(x)$ is

$$\int_0^A [f(x) - B]dx,$$

where the quantity demanded is A and the price is $B = f(A)$.

Definition The *future value of a continuous income stream* of K dollars per year for N years at interest rate r compounded continuously is

$$\int_0^N Ke^{r(N-t)}dt.$$

Definition The volume of the *solid of revolution* obtained from revolving the region below the graph of $y = g(x)$ from $x = a$ to $x = b$ about the x-axis is

$$\int_a^b \pi[g(x)]^2 dx.$$

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Problems from 11^{th} edition: $\#3, 5, 7, 10, 11, 13, 15, 16, 31, 32, 33.$

Determine the average value of $f(x)$ over the interval from $x = a$ to $x = b$, where:

3. $f(x) = e^{x/3}; a = 0,\ b = 3$

ANSWER:

$$\frac{1}{3-0}\int_0^3 f(x)dx = \frac{1}{3}\int_0^3 e^{x/3}dx = \frac{1}{3}\left[\frac{e^{x/3}}{1/3}\right]_{x=0}^3 \quad (1)$$

$$= e^{x/3}\,\Big|_{x=0}^3 \quad (2)$$

$$= e^{3/3} - e^{0/3} \quad (3)$$

$$= e^1 - e^0 = e - 1 \quad (4)$$

5. $f(x) = 1/x^2; a = \frac{1}{4},\ b = \frac{1}{2}$

ANSWER:

$$\frac{1}{1/2 - 1/4} \int_{1/4}^{1/2} \frac{1}{x^2}\, dx = \frac{1}{1/4} \left[\frac{x^{-1}}{-1} \right]_{x=1/4}^{1/2} \tag{5}$$

$$= 4 \left[-\frac{1}{x} \right]_{x=1/4}^{1/2} \tag{6}$$

$$= -4 \left[\frac{1}{x} \right]_{x=1/4}^{1/2} \tag{7}$$

$$= -4 \left[\frac{1}{1/2} - \frac{1}{1/4} \right] \tag{8}$$

$$= -4(2-4) = -4(-2) \tag{9}$$

$$= 8 \tag{10}$$

7. During a certain 12-hour period the temperature at time t (measured in hours from the start of the period) was $47 + 4t - \frac{1}{3}t^2$ degrees. What was the average temperature during that period?

ANSWER:

$$\frac{1}{12-0} \int_0^{12} \left(47 + 4t - \frac{1}{3}t^2 \right) dt = \tag{11}$$

$$= \frac{1}{12} \left[47t + 2t^2 - \frac{t^3}{9} \right]_{t=0}^{12} \tag{12}$$

$$= \frac{1}{12} \left[\left(47(12) + 2(12)^2 - \frac{(12)^3}{9} \right) - \left(47(0) + 2(0)^2 - \frac{(0)^3}{9} \right) \right] \tag{13}$$

$$= \frac{1}{12} \left[\left(47 + 2(12) - \frac{(12)^2}{9} \right) (12) \right] \tag{14}$$

$$= 47 + 24 - \frac{144}{9} = 47 + 24 - 16 \tag{15}$$

$$= 55 \tag{16}$$

10. One hundred dollars are deposited in the bank at 5% interest compounded continuously. What will be the average value of the money in the account during the next 20 years?

ANSWER: Recall the formula $P(t) = P_0 e^{rt}$. So given the above data, we appropriately substitute in the above values to obtain

$$P(t) = 100e^{.05t}.$$

Then

$$\frac{1}{20-0}\int_0^{20} P(t)dt = \frac{1}{20}\int_0^{20} 100e^{.05t}dt \tag{17}$$
$$= \frac{1}{20}(100)\frac{e^{.05t}}{.05}|_{t=0}^{20} \tag{18}$$
$$= \frac{10}{2}\left(\frac{1}{.05}\right)e^{.05t}|_{t=0}^{20} \tag{19}$$
$$= 5\left(\frac{1}{.05}\right)\left[e^{.05(20)} - e^0\right] \tag{20}$$
$$= \frac{500}{5}\left(e^1 - 1\right) \tag{21}$$
$$= 100(e-1) \tag{22}$$

Find the consumers' surplus for each of the following demand curves at the given sales level x.

11. $p = 3 - \frac{x}{10}; x = 20$
ANSWER: So $p = f(x) = 3 - \frac{x}{10}$. Since $x = 20$, $A = 20$ and
$B = f(A) = f(20) = 3 - \frac{20}{10} = 3 - 2 = 1$. So

$$\int_0^{20}\left[\left(3 - \frac{x}{10}\right) - 1\right]\,dx = \int_0^{20}\left(2 - \frac{x}{10}\right)\,dx \tag{23}$$
$$= \left[2x - \left(\frac{1}{10}\right)\left(\frac{x^2}{2}\right)\right]_{x=0}^{20} \tag{24}$$
$$= \left(2(20) - \frac{1}{10}\frac{20^2}{2}\right) - \left(2(0) - \frac{1}{10}(0)\right) \tag{25}$$
$$= 40 - \frac{400}{20} \tag{26}$$
$$= 40 - 20 \tag{27}$$
$$= 20 \tag{28}$$

13. $p = \frac{500}{x+10} - 3; x = 40$
ANSWER: Again, $p = f(x) = \frac{500}{x+10} - 3$, $A = 40$, and
$B = f(A) = f(40) = \frac{500}{50} - 3 = 10 - 3 = 7$. So

$$\int_0^{40} \left[\left(\frac{500}{x+10} - 3 \right) - 7 \right] \, dx = \int_0^{40} \left(\frac{500}{x+10} - 10 \right) \, dx \tag{29}$$

$$= [500 \ln |x+10| - 10x]_{x=0}^{40} \tag{30}$$

$$= [500 \ln(x+10) - 10x]_{x=0}^{40} \tag{31}$$

$$= (500 \ln(50) - 400) - (500 \ln(10) - 10(0)) \tag{32}$$

$$= 500 \ln(50) - 500 \ln 10 - 400 \tag{33}$$

$$= 500 \ln \left(\frac{50}{10} \right) - 400 \tag{34}$$

$$= 500 \ln(5) - 400 \tag{35}$$

15. See textbook for the figure. $p = .01x + 3$; $x = 200$

ANSWER: Before, we were working with consumers' surplus. Now, we're working with *(total) producers' surplus* when the quantity produced is A, the price is $B = g(A)$, and $p = g(x)$.

So $A = 200$, $B = g(A) = g(200) = .01(200) + 3 = 2 + 3 = 5$, and

$$\int_0^{200} (B - g(x)) \, dx = \int_0^{200} (5 - (.01x + 3)) \, dx \tag{36}$$

$$= \int_0^{200} (2 - .01x) \, dx \tag{37}$$

$$= \left[2x - .01 \frac{x^2}{2} \right]_{x=0}^{200} \tag{38}$$

$$= 2(200) - .01 \frac{200^2}{2} \tag{39}$$

$$= 400 - \frac{.01(40,000)}{2} \tag{40}$$

$$= 400 - \frac{400}{2} \tag{41}$$

$$= 400 - 200 \tag{42}$$

$$= 200 \tag{43}$$

16. See textbook for the figure. $p = \frac{x^2}{9} + 1; x = 3$

ANSWER: Again, we were working with consumers' surplus. Now we're working with *(total) producers' surplus* when the quantity produced is A, the price is $B = g(A)$, and $p = g(x)$.

So $A = 3$, $B = g(A) = g(3) = \frac{3^2}{9} + 1 = \frac{9}{9} + 1 = 2$, and

$$\int_0^3 (B - g(x))\, dx = \int_0^3 \left(2 - \left(\frac{x^2}{9} + 1\right)\right) dx \tag{44}$$

$$= \int_0^3 \left(1 - \frac{x^2}{9}\right) dx \tag{45}$$

$$= \left[x - \frac{x^3}{27}\right]_{x=0}^{3} \tag{46}$$

$$= 3 - \frac{3^3}{27} - (0 - 0) \tag{47}$$

$$= 3 - 1 = 2 \tag{48}$$

Find the volume of the solid of revolution generated by revolving about the x-axis the region under each of the following curves.

31. $y = kx$ from $x = 0$ to $x = h$ (generates a cone)

ANSWER:

$$\int_a^b \pi \left(g(x)\right)^2 dx = \int_0^h \pi (kx)^2 dx \tag{49}$$

$$= \int_0^h \pi k^2 x^2\, dx \tag{50}$$

$$= \pi k^2 \int_0^h x^2\, dx \tag{51}$$

$$= \pi k^2 \left[\frac{x^3}{3}\right]_{x=0}^{h} \tag{52}$$

$$= \pi k^2 \frac{h^3}{3} \tag{53}$$

32. $y = x^2$ from $x = 1$ to $x = 2$

ANSWER:

$$\int_1^2 \pi \left(x^2\right)^2 dx = \int_1^2 \pi x^4\, dx \tag{54}$$

$$= \pi \left[\frac{x^5}{5}\right]_{x=1}^{2} \tag{55}$$

$$= \pi \left[\frac{2^5}{5} - \frac{1}{5}\right] \tag{56}$$

$$= \pi \left(\frac{32}{5} - \frac{1}{5}\right) \tag{57}$$

$$= \pi \left(\frac{31}{5}\right) \tag{58}$$

33. $y = \sqrt{x}$ from $x = 0$ to $x = 4$ (The solid generated is called a *paraboloid*.)
ANSWER:

$$\int_0^4 \pi \left(\sqrt{x}\right)^2 \, dx = \int_0^4 \pi x \, dx \tag{59}$$

$$= \int_0^4 \pi x \, dx \tag{60}$$

$$= \pi \left[\frac{x^2}{2}\right]_{x=0}^4 \tag{61}$$

$$= \pi \left(\frac{4^2}{2} - \frac{0^2}{2}\right) \tag{62}$$

$$= \pi \frac{16}{2} \tag{63}$$

$$= 8\pi \tag{64}$$