Problems from 11^{th} edition: Section 7.1. $\#3, 4, 7, 10, 11, 15, 20, 23 - 26$.

3. Let $g(x,y,z) = \frac{x}{(y-z)}$. Compute $g(2,3,4)$ and $g(7,46,44)$.

SOLUTION:

$$g(2,3,4) = \frac{2}{3-4} = \frac{2}{-1} = -2$$

$$g(7,46,44) = \frac{7}{46-44} = \frac{7}{2}$$

4. Let $f(x,y,z) = x^2 e^{\sqrt{y^2+z^2}}$. Compute $f(1,0,1)$ and $f(2,3,-4)$.

SOLUTION:

$$f(1,0,1) = 1^2 e^{\sqrt{0^2+1^2}} = e^1 = e$$

$$f(2,3,-4) = 2^2 e^{\sqrt{3^2+(-4)^2}} = 4e^{\sqrt{9+16}} = 4e^{\sqrt{25}} = 4e^{\pm 5}$$

7. Find a formula $C(x,y,z)$ that gives the cost of materials for the closed rectangular box in Fig. 7(a), with dimensions in feet, assuming that the material for the top and bottom costs \$3 per square foot and the material for the sides costs \$5 per square foot.

SOLUTION: $C(x,y,z) = ?$

$$\text{Cost for top and bottom: } \$3(xy + xy) = 3(xy + xy) \quad (1)$$
$$= 3(2xy) = 6xy \quad (2)$$

$$\text{Cost for sides: } \$5(xz + yz + xz + yz) = 5(2xz + 2yz) \quad (3)$$
$$= 10xz + 10yz \quad (4)$$

$$\text{Cost for closed rectangular box :} \quad (5)$$
$$C(x,y,z) = 6xy + 10xz + 10yz \quad (6)$$

8. (For extra practice.) Find a formula $C(x,y,z)$ that gives the cost of material for the rectangular enclosure in Fig. 7(b), with dimensions in feet, assuming that the material for the top costs \$3 per square foot and the material for the back and two sides costs \$5 per square foot.

SOLUTION: $C(x,y,z) = ?$

$$\text{Cost for top: } \$3(xy) = 3xy \quad (7)$$

$$\text{Cost for back and two sides: } \$5(yz + xz + yz) = 5(xz + 2yz) \quad (8)$$
$$= 5xz + 10yz \quad (9)$$

$$\text{Cost for the rectangular enclosure :} \quad (10)$$
$$C(x, y, z) = 3xy + 5xz + 10yz \quad (11)$$

10. Let $f(x, y) = 10x^{2/5}y^{3/5}$. Show that $f(3a, 3b) = 3f(a, b)$.

SOLUTION:

$$\begin{aligned}
f(3a, 3b) &= 10(3a)^{2/5}(3b)^{3/5} && (12)\\
&= 10 \cdot 3^{2/5}a^{2/5}3^{3/5}b^{3/5} && (13)\\
&= 10 \cdot 3^{2/5}3^{3/5}a^{2/5}b^{3/5} && (14)\\
&= 3^{2/5}3^{3/5} \cdot 10a^{2/5}b^{3/5} && (15)\\
&= \left(3^2\right)^{1/5}\left(3^3\right)^{1/5} f(a, b) && (16)\\
&= \left(3^2 \cdot 3^3\right)^{1/5} f(a, b) && (17)\\
&= \left(3^5\right)^{1/5} f(a, b) && (18)\\
&= 3f(a, b) && (19)
\end{aligned}$$

11. The present value of A dollars to be paid t years in the future (assuming a 5% continuous interest rate) is $P(A, t) = Ae^{-.05t}$. Find and interpret $P(100, 13.8)$.

SOLUTION: $A = 100$ dollars is to be paid to the investor in $t = 13.8$ years at interest rate of 5% if the initial amount invested was $P(100, 13.8) = 100e^{-.05(13.8)}$.

NOTE: A, in this case, is the **future** amount, not initial amount because of the negative sign in front of $.05t$. We can derive back to our usual formula $A(t) = A_0e^{rt}$, where r is positive, using the following steps:

$$\begin{aligned}
P(A, t) &= Ae^{-.05t} && (20)\\
P(A, t)e^{.05t} &= \left(Ae^{-.05t}\right)e^{.05t} && (21)\\
P(A, t)e^{.05t} &= A && (22)
\end{aligned}$$

Draw the level curves of heights 0, 1, and 2 for the functions in Exercises 15 and 16.

15. f(x,y)=2x+y

SOLUTION:

$$\text{We first set} f(x, y) = 0 = 2x + y, \text{ which implies that } y = -2x.$$

Do you remember how to sketch the curve $y = -2x$ on a graph? This is a straight line passing through the origin with slope $m = -2$.

$$\text{Next we let } f(x, y) = 1 = 2x + y, \text{ which implies that } y = -2x + 1.$$

This line passes through the point $(0, 1)$ and has slope $m = -2$. This line is parallel to the first line that we have sketched when we had set $f(x, y) = 0$.

$$\text{Now we let } f(x, y) = 2 = 2x + y, \text{ which implies that } y = -2x + 2.$$

This line passes through the point $(0, 2)$ and has slope $m = -2$. This line is also parallel to the first line that which we have sketched when we had set $f(x, y) = 0$.

19. (For extra practice.) Find a function $f(x, y)$ that has the line $y = 3x - 4$ as a level curve.

SOLUTION:

$$y = 3x - 4 \Rightarrow y - 3x = -4$$

So $f(x, y) = y - 3x$.

20. Find a function $f(x, y)$ that has the curve $y = 3/x^2$ as a level curve.

SOLUTION:

$$y = \frac{3}{x^2} \Rightarrow x^2 y = 3$$

So $f(x, y) = x^2 y$.

Match the graphs of the functions in Exercises 23-26 to the systems of level curves shown in Figs. 8(a)-(d).

SOLUTION:

23. d.
24. b.
25. c.
26. a.

END OF SECTION 7.1.

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Problems from 11^{th} edition: Section 7.2. $\#3, 12, 15, 17, 19, 23, 24, 27, 30$.

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for each of the following functions.

3. $f(x, y) = 2x^2 e^y$

SOLUTION:

$$\frac{\partial f}{\partial x} = 2(2x)e^y = 4xe^y$$

$$\frac{\partial f}{\partial y} = 2x^2\left(e^y\right) = 2x^2 e^y$$

12. (#8. from 10^{th} edition.) $f(x, y) = \left(9x^2y + 3x\right)^{12}$

SOLUTION:

$$\text{Take the derivative with respect to } x \text{ and then use chain rule:} \tag{23}$$

$$\frac{\partial f}{\partial x} = 12\left(9x^2y + 3x\right)^{11}(18xy + 3) \tag{24}$$

$$\text{Take the derivative with respect to } y \text{ and then use chain rule:} \tag{25}$$

$$\frac{\partial f}{\partial y} = 12\left(9x^2y + 3x\right)^{11}\left(9x^2 + 0\right) \tag{26}$$

$$= 12\left(9x^2y + 3x\right)^{11}\left(9x^2\right) \tag{27}$$

(#12. from 10^{th} edition.) $f(x, y) = \frac{2xy}{e^x}$

SOLUTION:

$$\frac{\partial f}{\partial x} = \frac{e^x(2y) - 2xye^x}{e^{2x}} \tag{28}$$

$$= \frac{2ye^x}{e^{2x}} - \frac{2xye^x}{e^{2x}} \tag{29}$$

$$= \frac{2y}{e^x} - \frac{2xy}{e^x} \tag{30}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(f(x, y)) = \frac{\partial}{\partial y}\left(\frac{2x}{e^x}y\right) = \frac{2x}{e^x}$$

15. Let $f(x, y, z) = \left(1 + x^2y\right)/z$. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.

SOLUTION:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{z} + \frac{x^2y}{z}\right) \tag{31}$$

$$= 0 + \frac{2xy}{z} = \frac{2xy}{z} \tag{32}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{z} + \frac{x^2 y}{z}\right) \tag{33}$$
$$= 0 + \frac{x^2}{z} = \frac{x^2}{z} \tag{34}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\left(1 + x^2 y\right)(z)^{-1}\right) \tag{35}$$
$$= -\left(1 + x^2 y\right) z^{-2} \tag{36}$$
$$= -\frac{1 + x^2 y}{z^2} \tag{37}$$

17. (#18. from 10^{th} edition.) Let $f(x, y, z) = ze^{(x+3y)z}$. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.

SOLUTION: $f(x, y, z) = ze^{(x+3y)z} = ze^{xz+3yz}$

$$\frac{\partial f}{\partial x} = ze^{xz+3yz} \cdot (z + 0) \tag{38}$$
$$= ze^{xz+3yz} \cdot (z) \tag{39}$$
$$= z^2 e^{xz+3yz} \tag{40}$$

$$\frac{\partial f}{\partial y} = ze^{xz+3yz} \cdot (3z) \tag{41}$$

Use product rule:

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(ze^{(x+3y)z}\right) \tag{42}$$
$$= z\left(e^{(x+3y)z}(x + 3y)\right) + e^{(x+3y)z} \tag{43}$$
$$= e^{(x+3y)z}(zx + 3zy + 1) \tag{44}$$

19. Let $f(x, y) = x^2 + 2xy + y^2 + 3x + 5y$. Find $\frac{\partial f}{\partial x}(2, -3)$ and $\frac{\partial f}{\partial y}(2, -3)$.

SOLUTION: Compute the first order partial and then plug in at the point $(x, y) = (2, -3)$:

$$\frac{\partial f}{\partial x} = 2x + 2y + 3$$
$$\frac{\partial f}{\partial x}(2, -3) = 4 - 6 + 3 = 1$$

and

$$\frac{\partial f}{\partial y} = 2x + 2y + 5$$

$$\frac{\partial f}{\partial y}(2, -3) = 4 - 6 + 5 = 3.$$

23. Let $f(x, y) = x^3y + 2xy^2$. Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$.
SOLUTION: Compute the first order partials:

$$\frac{\partial f}{\partial x} = 3x^2y + 2y^2$$

$$\frac{\partial f}{\partial y} = x^3 + 4xy.$$

Now we compute the second order partials:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \quad (45)$$

$$= \frac{\partial}{\partial x}\left(3x^2y + 2y^2\right) \quad (46)$$

$$= 6xy \quad (47)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \quad (48)$$

$$= \frac{\partial}{\partial y}\left(x^3 + 4xy\right) \quad (49)$$

$$= 4x \quad (50)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad (51)$$

$$= \frac{\partial}{\partial x}\left(x^3 + 4xy\right) \quad (52)$$

$$= 3x^2 + 4y \quad (53)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \quad (54)$$

$$= \frac{\partial}{\partial y}\left(3x^2y + 2y^2\right) \quad (55)$$

$$= 3x^2 + 4y \quad (56)$$

24. Let $f(x,y) = xe^y + x^4y + y^3$. Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$.

SOLUTION: Compute the first order partials:

$$\frac{\partial f}{\partial x} = e^y + 4x^3y$$

$$\frac{\partial f}{\partial y} = xe^y + x^4 + 3y^2$$

Now we compute the second order partials:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \tag{57}$$

$$= \frac{\partial}{\partial x}\left(e^y + 4x^3y\right) \tag{58}$$

$$= 12x^2y \tag{59}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \tag{60}$$

$$= \frac{\partial}{\partial y}\left(xe^y + x^4 + 3y^2\right) \tag{61}$$

$$= xe^y + 6y \tag{62}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \tag{63}$$

$$= \frac{\partial}{\partial x}\left(xe^y + x^4 + 3y^2\right) \tag{64}$$

$$= e^y + 4x^3 \tag{65}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \tag{66}$$

$$= \frac{\partial}{\partial y}\left(e^y + 4x^3y\right) \tag{67}$$

$$= e^y + 4x^3 \tag{68}$$

27. In a certain suburban community, commuters have the choice of getting into the city by bus or train. The demand for these modes of transportation varies with their cost. Let $f(p_1, p_2)$ be the number of people who will take the bus when p_1 is the price of the bus ride and p_2 is the price of the train ride. For example, if $f(4.50, 6) = 7000$, then 7000 commuters will take the bus when

the price of a bus ticket is \$4.50 and the price of a train ticket is \$6.00. Explain why $\frac{\partial f}{\partial p_1} < 0$ and $\frac{\partial f}{\partial p_2} > 0$.

SOLUTION:

$$f(p_1, p_2) = \#\text{ of people who will take the bus} \tag{69}$$
$$\text{when } p_1 \text{ is the price of the bus ride} \tag{70}$$
$$\text{and } p_2 \text{ is the price of the train ride.} \tag{71}$$

$$p_1 = \text{ the price of the bus ride}$$

$$p_2 = \text{ the price of the train ride}$$

If the price of a bus ride p_1 increases and the price of a train ticket p_2 remains constant, fewer people will ride the bus.

An increase in train-ticket prices p_2 coupled with constant bus fare p_1 should cause more people to ride the bus.

30. The demand for a certain gas-guzzling car is given by $f(p_1, p_2)$, where p_1 is the price of the car and p_2 is the price of gasoline. Explain why $\frac{\partial f}{\partial p_1} < 0$ and $\frac{\partial f}{\partial p_2} < 0$.

SOLUTION:

$$f(p_1, p_2) = \text{ the demand for a certain gas-guzzling car}$$

$$p_1 = \text{ the price of the gas-guzzling car}$$

$$p_2 = \text{ the price of gasoline}$$

If the price of the gas-guzzling car p_1 increases and the price of gasoline p_2 remains constant, then fewer people will buy the gas-guzzling car.

If the price of gasoline p_2 increases and the price of gas-guzzling car p_1 remains constant, then fewer people will buy the gas-guzzling car.

END OF SECTION 7.2.

Please submit all corrections to mim2 (at) math.uiuc.edu.