Let X be a Riemann surface and let $\mathcal{U} = \{U : U \text{ open in } X\}$ be an open cover of X. Then E is a line bundle with the map $\Psi : E \to X$ so that for all $u \in U$, the diagram

$$E \supseteq \Psi^{-1}(U) \longleftarrow U \times \mathbb{C} : \Psi_U \quad (1)$$
$$\Psi \searrow \quad \swarrow P \quad (2)$$
$$U \quad (3)$$

commutes, where $\Psi_U : U \times \mathbb{C} \to \Psi^{-1}(U)$ is an isomorphism and $g_{UV} : U \cap V \to GL(\mathbb{C}, 1) \cong \mathbb{C}^*$ are transition functions for E given by

$$\Psi_U(p, v) = \Psi_V(p, g_{UV}(p)v)$$

where $p \in U \cap V$ so that the maps $\{g_{UV}\}$ satisfy the cycle conditions

$$g_{VU} = g_{UV}^{-1} \text{ and } g_{UW} = g_{UV} \circ g_{VW}.$$

We say two line bundles E and E' are equivalent if there are functions $\{h_U\}_{U \in \mathcal{U}}$ so that $h_U \circ g_{UV} \circ h_V^{-1} = g'_{UV}$

Let us define $\mathcal{A} = \{\text{germs of } C^\infty\text{-functions on } X\}$ and
$\mathcal{A}^* = \{\text{germs of nonvanishing } C^\infty\text{-functions on } X\}$. Then

$$H^1(X, \mathcal{A}^*) = \frac{\text{kernel}}{\text{image}} = \frac{Z^1}{B^1} \quad (4)$$
$$= \frac{\{g_{UV} : g_{UV} \circ g_{VW} \circ g_{WU} = 1\}}{\{g_{UV} = h_V / h_U\}} \quad (5)$$

with

$$g_{UV} \sim g'_{UV} \Leftrightarrow \frac{g_{UV}}{g'_{UV}} = \frac{h_V}{h_U} \in B^1.$$

Example. Consider the short exact sequence

$$0 \to \mathbb{Z} \hookrightarrow \mathcal{A} \twoheadrightarrow \mathcal{A}^* \to 0$$

where the map from $\mathcal{A}$ to $\mathcal{A}^*$ is $exp(2\pi i) : \mathcal{A} \to \mathcal{A}^*$. This induces a long exact sequence of cohomology but for now, we're interested in

$$0 = H^1(X, \mathcal{A}) \to H^1(X, \mathcal{A}^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{A}) = 0$$

part of the long exact sequence. Note that since $H^q(X, \mathcal{A}) = 0$ for $q > 0$, we have $H^1(X, \mathcal{A}) = H^2(X, \mathcal{A}) = 0$. So $H^1(X, \mathcal{A}^*) \cong H^2(X, \mathbb{Z})$. For X compact, $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$. Thus, for a compact Riemann surface, $Pic(X) = H^1(X, \mathcal{A}^*) \cong \mathbb{Z}$.

Definition. *We define the (first) Chern class of a line bundle E to be c(E) where* $c : H^1(X, \mathcal{A}^*) \to H^2(X, \mathbb{Z})$.

A consequence of the definition is that $c(E \otimes E') = c(E) + c(E')$ and $c(E^*) = c(E)$.

Example. Let $X = \mathbb{P}^1$. Then $c : TX \to 1$ and $T^*X \to -1 \in H^2(X, \mathbb{Z})$.

Let $U = \mathbb{P}^1 \backslash \{\infty\}$ and $V = \mathbb{P}^1 \backslash \{0\}$ be an open cover of $\mathbb{P}^1$. Then $\psi_U : U \to \mathbb{C}$ where $\psi_U(z) = z$, $\psi_V : V \to \mathbb{C}$ where $\psi_V(z) = 1/z$, and $c(g_{UV}) = g_{UVW} = g_{VW} - g_{UW} + g_{UV}$ for $[g_{UV}] \in H^1(X, \mathcal{O}^*)$.

Then

$$\overline{\partial} \log \overline{g_{UV}} = \frac{1}{\overline{g_{UV}}} \overline{g_{UV}}' = 0$$

and $r_U = |g_{UV}|^2 r_V$. So

$$c([g_{UV}]) = \iint_X \eta$$

where $\eta := \frac{1}{2\pi i} \partial \overline{\partial} \log r_U = \frac{1}{2\pi i} \left(\partial \overline{\partial} \log |g_{UV}|^2 + \partial \overline{\partial} \log r_V \right)$ and we see that $\partial \overline{\partial} \log |g_{UV}|^2 = \partial \overline{\partial} \log (g_{UV} \overline{g_{UV}}) = \partial \overline{\partial} \log g_{UV} + \partial \overline{\partial} \log \overline{g_{UV}}$.

Example. Now consider the exact sheaf sequence $0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0$, where $\mathcal{O}^*$ are nonvanishing holomorphic functions on X, $\mathcal{M}^*$ are nonvanishing meromorphic functions on X, and $\mathcal{D}$ are divisors on X. Then

$$\frac{\mathcal{M}_p^*}{\mathcal{O}_p^*} = \left\{ \sum n_p [p] \right\}.$$

Let $A(X) = \{\text{divisors}\}/\sim$ where $\mathcal{D} \sim \mathcal{D}'$ if and only if $\mathcal{D} = \mathcal{D}' + (f)$ for some $f \in \mathcal{M}^*(X)$. So $0 \to A(X) \to H^1(\mathcal{O}^*) \to H^1(\mathcal{M}^*) = 0 \to 0$ imply $A(X) \cong H^1(\mathcal{O}^*)$.

Recall that $\mathcal{O}(E) = \Gamma(E) =$ holomorphic sections of line bundle E, and $\mathcal{O}(K) \cong \mathcal{O}^{1,0}(X)$ where K is canonical line bundle.

Using Riemann-Roch, $\dim H^0(X, \mathcal{O}(E)) - \dim H^1(X, \mathcal{O}(E)) - c(E) = 1 - g$. Since $\dim H^0(X, \mathcal{O}(K)) = g$ and $\dim H^1(X, \mathcal{O}(K)) = 1$, we conclude $c(K) = 2g - 2$.

A final remark. Let X be a compact Riemann surface. Then we have

$$\iint_X \eta = \frac{1}{2\pi i} \iint_X \partial \overline{\partial} \log r_U \tag{6}$$

$$= \frac{1}{2\pi i} \iint_X d \overline{\partial} \log r_U \tag{7}$$

$$= \frac{1}{2\pi i} \int_{\partial X} \overline{\partial} \log r_U \text{ by Green's theorem} \tag{8}$$

$$= \frac{1}{2\pi i} \int_{\varnothing} \overline{\partial} \log r_U = 0. \tag{9}$$

Do you see a mistake in this computation?

Yes, one should see a mistake because $\frac{1}{2\pi i}\iint_X d\overline{\partial}\log r_U \neq \frac{1}{2\pi i}\int_{\partial X}\overline{\partial}\log r_U$. While $d\overline{\partial}\log r_U$ is globally defined on X, $\overline{\partial}\log r_U$ is not globally defined. Thus we can't apply Green's Theorem to the above computation.