1. Sketch the graph of a function with the following properties:
The points (0,6), (2,3), and (4,0) are on the graph,
$f'(0) = 0, f'(4) = 0,$
$f''(x) < 0$ for $x < 2$,
$f''(x) > 0$ for $x > 2$,
$f''(2) = 0.$
ANSWER: See your TA for help if you need it.

2. A rectangular corral of 65 square meters is to be fensed off and then divided by two fences into three sections. Suppose the cost of fencing for the boundary is 5 dollars per meter and the dividing fences cost 3 dollars per meter.

a. Suppose we would like to minimize the cost of the fence under these conditions. Find the objective equation.
ANSWER: $C(x, y) = C = 5(2x + 2y) + 3(2y) = 10x + 16y$
b. Determine the constraint.
ANSWER: $A = xy = 65$
c. Use the constraint to write the objective equation as a function of ONE variable.
ANSWER: Since $y = 65/x$, $C = 10x + 16\frac{65}{x}$.

3. Given that x and y are related by the following equation, answer the questions below. Show your work.

$$x^3y^2 + 5x^2 = 7y + 21$$

a. Find $\frac{dy}{dx}$.
ANSWER:

$$\frac{d}{dx}(x^3y^2) + \frac{d}{dx}(5x^2) = \frac{d}{dx}(7y) + \frac{d}{dx}(21) \quad (1)$$

$$3x^2y^2 + 2x^3y\frac{dy}{dx} + 10x = 7\frac{dy}{dx} + 0 \quad (2)$$

$$3x^2y^2 + 10x = (7 - 2x^3y)\frac{dy}{dx} \quad (3)$$

$$\frac{3x^2y^2 + 10x}{7 - 2x^3y} = \frac{dy}{dx} \quad (4)$$

b. Determine an equation of the line tangent to this curve at the point $(2, 1)$.
ANSWER: Since we have found dy/dx (which is the slope), we only need to evaluate dy/dx at the given point $(2, 1)$. So we have

$$m = \frac{dy}{dx}|_{(2,1)} = \frac{3(4)(1) + 10(2)}{7 - 2(8)(1)} = \frac{12 + 20}{7 - 16} = \frac{32}{-9}.$$

Thus the equation of the line tangent to this curve is

$$y - 1 = -\frac{32}{9}(x - 2).$$

4. Answer the questions below. Show your work.
a. Given $f(x) = \ln(x^2 + 3x + 1) + \ln 4 + x^2 \ln(4x)$, find $f'(x)$.
ANSWER: Use chain rule.

$$f'(x) = \frac{2x + 3}{x^2 + 3x + 1} + 0 + x^2 \frac{4}{4x} + 2x \ln(4x)$$

b. Given $g(t) = (e^{t^2+1} + 4)^5 + e$, find $g'(t)$.
ANSWER: Use chain rule.

$$g'(t) = 5(e^{t^2+1} + 4)^4[(e^{t^2+1})(2t) + 0] + 0$$

c. Solve the following equation for x: $e^{4+3\ln(x)} = 5$.
ANSWER:
METHOD 1.

$$e^{4+3\ln(x)} = 5 \quad (5)$$
$$\ln e^{4+3\ln(x)} = \ln 5 \quad (6)$$
$$4 + 3\ln(x) = \ln 5 \quad (7)$$
$$3\ln(x) = \ln 5 - 4 \quad (8)$$
$$\ln(x) = \frac{\ln 5 - 4}{3} \quad (9)$$
$$e^{\ln(x)} = e^{\frac{\ln 5 - 4}{3}} \quad (10)$$
$$x = e^{\frac{\ln 5 - 4}{3}} = e^{\frac{1}{3}(\ln 5 - 4)} \quad (11)$$

METHOD 2.

$$e^{4+3\ln(x)} = 5 \quad (12)$$
$$e^4 e^{3\ln(x)} = 5 \quad (13)$$
$$e^4 e^{(\ln(x))^3} = 5 \quad (14)$$
$$e^4 x^3 = 5 \quad (15)$$
$$x^3 = \frac{5}{e^4} \quad (16)$$
$$x = \left(\frac{5}{e^4}\right)^{\frac{1}{3}} \quad (17)$$

5. A toll road averages 1000 cars per day when charging 50 cents per car. A survey concludes the toll will result in 10 fewer cars for each 1 cent increase in price.

a. Determine the toll company's **revenue function** in terms of the toll price.

ANSWER: Let x = price increase in cents (we're starting out with a minimum of 50 cents) and let p = number of cars. Then

$$R(x) = p(50 + x) \tag{18}$$
$$= (1000 - 10x)(50 + x) \tag{19}$$

b. Determine the toll price that will maximize the toll company's revenue. **Be sure to check that you have found the max.**

ANSWER: Apply chain rule here to compute the derivative of $R(x)$.

$$R'(x) = (1000 - 10x)1 + (-10)(50 + x) \tag{20}$$
$$= 1000 - 10x - 500 - 10x \tag{21}$$
$$= 500 - 20x \tag{22}$$

In order to find the local maximum, we need to set $R'(x)$ equal to zero. So

$$R'(x) = 500 - 20x = 0 \tag{23}$$
$$500 = 20x \tag{24}$$
$$\frac{500}{20} = x \tag{25}$$
$$25 = x \tag{26}$$

So is this a local max or a local min? In order to figure this out, compute $R''(x)$ and let $x = 25$. If $R''(25) < 0$, the graph is concave down. So it's a local max. If $R''(25) > 0$, the graph is concave up and this is a local min. Now computing $R''(x)$,

$$R''(x) = -20 \tag{27}$$
$$R''(25) = -20 < 0, \tag{28}$$

we see that $x = 25$ is a local max and the toll price should be $50 + 25 = 75$ cents.

6. Use **logarithmic differentiation** to differentiate the following function. Your answer should be in terms of x only. Show your work.

$$f(x) = \frac{\sqrt{x - 4}x^5 e^{2x}}{(x + 7)^4}$$

ANSWER:

$$\ln f(x) = \ln\left(\frac{\sqrt{x-4}x^5e^{2x}}{(x+7)^4}\right) \tag{29}$$
$$\ln f(x) = \ln(\sqrt{x-4}) + \ln(x^5) + \ln(e^{2x}) - \ln((x+7)^4) \tag{30}$$
$$\ln f(x) = \ln(x-4)^{1/2} + \ln(x)^5 + \ln(e)^{2x} - \ln(x+7)^4 \tag{31}$$
$$\ln f(x) = \frac{1}{2}\ln(x-4) + 5\ln(x) + (2x)\ln(e) - 4\ln(x+7) \tag{32}$$
$$\ln f(x) = \frac{1}{2}\ln(x-4) + 5\ln(x) + (2x)1 - 4\ln(x+7) \tag{33}$$

Now differentiate both sides.

$$\frac{d}{dx}(\ln f(x)) = \frac{1}{2}\frac{1}{x-4} + 5\frac{1}{x} + 2 - 4\frac{1}{x+7}.$$

We know from earlier this semester that

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}.$$

Thus it follows that

$$\frac{f'(x)}{f(x)} = \frac{1}{2}\frac{1}{x-4} + 5\frac{1}{x} + 2 - 4\frac{1}{x+7}$$

and we conclude

$$f'(x) = \left(\frac{\sqrt{x-4}x^5e^{2x}}{(x+7)^4}\right)\left(\frac{1}{2(x-4)} + \frac{5}{x} + 2 - \frac{4}{x+7}\right).$$