

1. Sketch the graph of a function with the following properties:

The points $(-3,0)$, $(0,2)$, and $(3,4)$ are on the graph,

$$f'(-3) = 0, f'(3) = 0,$$

$$f''(x) > 0 \text{ for } x < 0,$$

$$f''(x) < 0 \text{ for } x > 0,$$

$$f''(0) = 0.$$

ANSWER: See your TA for help if you need it.

2. A rectangular corral of 54 square meters is to be fenced off and then divided by a fence into two sections. Suppose the cost of fencing for the boundary is 6 dollars per meter and the dividing fence costs 3 dollars per meter.

a. Suppose we would like to minimize the cost of the fence under these conditions. Find the objective equation.

$$\text{ANSWER: } C(l, w) = C = 6(2l + 2w) + 3(w) = 12l + 15w$$

b. Determine the constraint.

$$\text{ANSWER: } A = lw = 54$$

c. Use the constraint to write the objective equation as a function of ONE variable.

$$\text{ANSWER: Since } l = 54/w, C = 12\frac{54}{w} + 15w.$$

3. Given that x and y are related by the following equation, answer the questions below. Show your work.

$$4x - x^2y^3 = 6 - 5y$$

a. Find $\frac{dy}{dx}$.

ANSWER:

$$\frac{d}{dx}(4x) - \frac{d}{dx}(x^2y^3) = \frac{d}{dx}(6) - \frac{d}{dx}(5y) \quad (1)$$

$$4 - 2xy^3 - 3x^2y^2\frac{dy}{dx} = 0 - 5\frac{dy}{dx} \quad (2)$$

$$4 - 2xy^3 = (3x^2y^2 - 5)\frac{dy}{dx} \quad (3)$$

$$\frac{4 - 2xy^3}{3x^2y^2 - 5} = \frac{dy}{dx} \quad (4)$$

b. Determine an equation of the line tangent to this curve at the point $(2, 1)$.

ANSWER: Since we have found dy/dx (which is the slope), we only need to evaluate dy/dx at the given point $(1, 2)$. So we have

$$m = \left. \frac{dy}{dx} \right|_{(1,2)} = \frac{4 - 2(1)(8)}{3(1)(4) - 5} = \frac{4 - 16}{12 - 5} = \frac{-12}{7}.$$

Thus the equation of the line tangent to this curve is

$$y - 2 = -\frac{12}{7}(x - 1).$$

4. Answer the questions below. Show your work.

a. Given $f(x) = e^{3x^2+5x+1} + e + x^2e^{7x}$, find $f'(x)$.

ANSWER: Use chain rule.

$$f'(x) = e^{3x^2+5x+1}(6x + 5) + 0 + x^2e^{7x}(7) + 2xe^{7x}$$

b. Given $g(t) = (\ln(4t + 2))^3 + \ln 2$, find $g'(t)$.

ANSWER: Use chain rule.

$$g'(t) = 3(\ln(4t + 2))^2 \frac{4}{4t + 2} + 0$$

c. Solve the following equation for x: $\ln(3 + 2\ln(x)) = 4$.

ANSWER:

$$\ln(3 + 2\ln(x)) = 4 \tag{5}$$

$$e^{\ln(3+2\ln(x))} = e^4 \tag{6}$$

$$3 + 2\ln(x) = e^4 \tag{7}$$

$$2\ln(x) = e^4 - 3 \tag{8}$$

$$\ln(x) = \frac{e^4 - 3}{2} \tag{9}$$

$$e^{\ln(x)} = e^{\frac{e^4-3}{2}} \tag{10}$$

$$x = e^{\frac{1}{2}(e^4-3)} \tag{11}$$

5. The Calculus-For-Profit company is offering a tour of Altgeld Hall and exclusive front row seats to Calculus lectures. The price for a group of 10 students is 1000 dollars per student; but for each additional student above the 10-student minimum, the cost is reduced by 20 dollars per student.

a. Determine the company's **revenue function** in terms of the number of students.

ANSWER: Let x = additional students (we're starting out with a minimum of 10 students) and let p = price paid per student. Then

$$R(x) = p(10 + x) \tag{12}$$

$$= (1000 - 20x)(10 + x) \tag{13}$$

b. Determine the number of students that will maximize the company's revenue. **Be sure to check that you have found the max.**

ANSWER: Apply chain rule here to compute the derivative of $R(x)$.

$$R'(x) = (1000 - 20x)1 + (-20)(10 + x) \quad (14)$$

$$= 1000 - 20x - 200 - 20x \quad (15)$$

$$= 800 - 40x \quad (16)$$

In order to find the local maximum, we need to set $R'(x)$ equal to zero. So

$$R'(x) = 800 - 40x = 0 \quad (17)$$

$$800 = 40x \quad (18)$$

$$\frac{800}{40} = x \quad (19)$$

$$20 = x \quad (20)$$

So is this a local max or a local min? In order to figure this out, compute $R''(x)$ and let $x = 20$. If $R''(20) < 0$, the graph is concave down. So it's a local max. If $R''(20) > 0$, the graph is concave up and this is a local min. Now computing $R''(x)$,

$$R''(x) = -40 \quad (21)$$

$$R''(20) = -40 < 0, \quad (22)$$

we see that $x = 20$ is a local max and the number of students to maximize the company's revenue is $10 + 20 = 30$.

6. Use **logarithmic differentiation** to differentiate the following function. Your answer should be in terms of x only. Show your work.

$$f(x) = \frac{e^x(x-2)^3x^4}{\sqrt{x+5}}$$

ANSWER:

$$\ln f(x) = \ln \left(\frac{e^x(x-2)^3x^4}{\sqrt{x+5}} \right) \quad (23)$$

$$\ln f(x) = \ln(e^x) + \ln((x-2)^3) + \ln(x^4) - \ln(\sqrt{x+5}) \quad (24)$$

$$\ln f(x) = \ln(e)^x + \ln(x-2)^3 + \ln(x)^4 - \ln(x+5)^{1/2} \quad (25)$$

$$\ln f(x) = x \ln(e) + 3 \ln(x-2) + 4 \ln(x) - \frac{1}{2} \ln(x+5) \quad (26)$$

$$\ln f(x) = x(1) + 3 \ln(x-2) + 4 \ln(x) - \frac{1}{2} \ln(x+5) \quad (27)$$

Now differentiate both sides.

$$\frac{d}{dx}(\ln f(x)) = 1 + 3 \frac{1}{x-2} + 4 \frac{1}{x} - \frac{1}{2} \frac{1}{x+5}.$$

We know from earlier this semester that

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}.$$

Thus it follows that

$$\frac{f'(x)}{f(x)} = 1 + 3\frac{1}{x-2} + 4\frac{1}{x} - \frac{1}{2}\frac{1}{x+5}$$

and we conclude

$$f'(x) = \left(\frac{e^x(x-2)^3x^4}{\sqrt{x+5}} \right) \left(1 + \frac{3}{x-2} + \frac{4}{x} - \frac{1}{2(x+5)} \right).$$