1. The group D_4 is the group of symmetries of the square in $\mathbb{R}^2$. It has generators ψ, (a reflection through the x-axis) and ρ (a counter-clockwise rotation by $\pi/2$). It acts on the natural basis $(\mathbf{e_1}, \mathbf{e_2})$ of $\mathbb{R}^2$.

(a) Write down the resulting 2-dimensional representation of D_4 by expressing the matrix action of the elements of D_4 on the natural basis.

(b) Group the elements of D_4 into conjugacy classes, and write down the trace of each of the matrices in your representation. Do you notice a correspondence?

Solution.

(a) Consider the group homomorphism $\pi_2 : D_4 \rightarrow Aut(\mathbb{R}^2) \cong GL_2(\mathbb{R})$. The order of D_4 is 8 with $D_4 = \{e, \rho, \rho^2, \rho^3, \psi, \psi\rho, \psi\rho^2, \psi\rho^3\}$. Then

$$\pi_2(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ since } e \text{ does not permute the basis,}$$

$$\pi_2(\rho) = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ since } \rho \text{ takes } e_1 \text{ to } e_2 \text{ and } e_2 \text{ to -}e_1,$$

$$\pi_2(\rho^2) = \begin{bmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\pi_2(\rho^3) = \begin{bmatrix} \cos(3\pi/2) & -\sin(3\pi/2) \\ \sin(3\pi/2) & \cos(3\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\pi_2(\psi) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ since } \psi \text{ is a reflection about the } x\text{-axis,}$$

$$\pi_2(\rho\psi) = \pi_2(\rho)\pi_2(\psi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\pi_2(\rho^2\psi) = \pi_2(\rho)^2\pi_2(\psi) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and}$$

$$\pi_2(\rho^3\psi) = \pi_2(\rho)^3\pi_2(\psi) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

(b) The group D_4 has 5 conjugacy classes: $\mathcal{O}_e = \{e\}$, $\mathcal{O}_\rho = \{\rho, \rho^3\}$, $\mathcal{O}_{\rho^2} = \{\rho^2\}$, $\mathcal{O}_{\rho\psi} = \{\rho\psi, \rho^3\psi\}$, and $\mathcal{O}_\psi = \{\psi, \rho^2\psi\}$. Then $\text{tr}(\pi_2(e)) = 2$, $\text{tr}(\pi_2(\rho)) = \text{tr}(\pi_2(\rho^3)) = 0$, $\text{tr}(\pi_2(\rho^2)) = -2$, $\text{tr}(\pi_2(\rho\psi)) = \text{tr}(\pi_2(\rho^3\psi)) = 0$, and $\text{tr}(\pi_2(\psi)) = \text{tr}(\pi_2(\rho^2\psi)) = 0$. From this particular group representation, we conclude that when two elements are in the same orbit, their traces are the same.

2. The group S_3 is the group of permutations of 3 elements.

(a) There is a natural action of S_3 on $\mathbb{R}^3$ by permutations of the basis elements $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$. Write down the matrix representation resulting from this action.

(b) There is an action on the two-dimensional subspace of $\mathbb{R}^3$ spanned by $(\mathbf{e_1} - \mathbf{e_2}, \mathbf{e_2} - \mathbf{e_3})$. Write down the resulting two-dimensional representation.

(c) Write down the representation on the one-dimensional subspace with basis $\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$.

(d) For each of the representation above, group the elements of S_3 into conjugacy classes and write down the trace of the matrix corresponding to the elements in each conjugacy class in each representation.

Solution.

(a) Since S_3 is generated by a two cycle and a three cycle, for instance, (12) and (123), it suffices to write down the matrix representation for the generators. So consider the group homomorphism $\pi_3 : S_3 \to GL_3(\mathbb{R})$. The element $(12) \in S_3$ interchanges $\mathbf{e_1}$ and $\mathbf{e_2}$. So

$$\pi_3((12)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As for $((123)) \in S_3$, it takes $\mathbf{e_1}$ to $\mathbf{e_2}$, $\mathbf{e_2}$ to $\mathbf{e_3}$, and $\mathbf{e_3}$ to $\mathbf{e_1}$. Thus

$$\pi_3((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since π_3 is a homomorphism for any σ, $\tau \in S_3$, $\pi_3(\sigma \cdot \tau) = \pi_3(\sigma) \cdot \pi_3(\tau)$.

(b) Let $\pi_2 : S_3 \to GL_2(\mathbb{R})$ be a group homomorphism. Let $\mathbf{v_1} = \mathbf{e_1} - \mathbf{e_2}$ and $\mathbf{v_2} = \mathbf{e_2} - \mathbf{e_3}$. Then S_3 acts on $Span_{\mathbb{R}}(v_1, v_2)$ in the following way:

The element (12) in the permutation group takes $\mathbf{e_1}$ to $\mathbf{e_2}$ and $\mathbf{e_2}$ to $\mathbf{e_1}$ while fixing $\mathbf{e_3}$. Thus $(12) \cdot \mathbf{v_1} = \mathbf{e_2} - \mathbf{e_1} = -\mathbf{v_1}$ while $(12) \cdot \mathbf{v_2} = \mathbf{e_1} - \mathbf{e_3} = \mathbf{v_1} + \mathbf{v_2}$. Thus

$$\pi_2((12)) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

As for the 3-cycle (123), it takes $\mathbf{e_1}$ to $\mathbf{e_2}$, $\mathbf{e_2}$ to $\mathbf{e_3}$, and $\mathbf{e_3}$ to $\mathbf{e_1}$. So $(123) \cdot \mathbf{v_1} = \mathbf{e_2} - \mathbf{e_3} = \mathbf{v_2}$ and $(123) \cdot \mathbf{v_2} = \mathbf{e_3} - \mathbf{e_1} = -\mathbf{v_1} - \mathbf{v_2}$. So

$$\pi_2((123)) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

(c) Let $\mathbf{w} = \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$. Then (12) takes $\mathbf{w}$ to $\mathbf{e_2} + \mathbf{e_1} + \mathbf{e_3}$ which is precisely $\mathbf{w}$. The element (123) also takes $\mathbf{w}$ to itself. So for all $\sigma \in S_3$, $\pi_1(\sigma) = [1] \in GL_1(\mathbb{R})$.

(d) The group S_3 has three conjugacy classes: $\mathcal{O}_e = \{e\}$, $\mathcal{O}_{(12)} = \{(12), (13), (23)\}$, and $\mathcal{O}_{(123)} = \{(123), (132)\}$.

Consider part 2(a). Then

$$\pi_3(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\pi_3((12)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \pi_3((13)) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \pi_3((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\pi_3((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \pi_3((132)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then $\mathrm{tr}(\pi_3(e)) = 3$, $\mathrm{tr}(\pi_3((12)))$=$\mathrm{tr}(\pi_3((13)))$=$\mathrm{tr}(\pi_3((23))) = 1$ and $\mathrm{tr}(\pi_3((123)))$ $=$ $\mathrm{tr}(\pi_3((132)))$=0.

Next consider part 2(b). Then

$$\pi_2(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\pi_2((12)) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \pi_2((13)) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \pi_2((23)) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix},$$

$$\pi_2((123)) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \pi_2((132)) = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

So $\mathrm{tr}(\pi_2(e))$=2, $\mathrm{tr}(\pi_2((12))) = \mathrm{tr}(\pi_2((13)))$= $\mathrm{tr}(\pi_2((23)))$= 0, and $\mathrm{tr}(\pi_2((123)))$=$\mathrm{tr}(\pi_2((132)))$= -1.

Finally consider part 2(c). Then for any $\sigma \in S_3$, $\mathrm{tr}(\pi_1(\sigma))$=1.

3. The group S_3 acts on itself by left multiplication. Write down the resulting six-dimensional representation of the two generators (12) and (123).

Solution. Consider $S_3 =< (12), (123) >= \{e, (12), (13), (23), (123), (132)\}$. Let $r_1 = e$, $r_2 = (12)$, $r_3 = (13)$, $r_4 = (23)$, $r_5 = (123)$, $r_6 = (132)$. Then since $(12)e = (12)$, $(12)(12) = e$, $(12)(13) = (132)$, $(12)(23) = (123)$, $(12)(123) = (23)$, and $(12)(132) = (13)$, for $\pi_6 : S_3 \to Aut(S_3)$

$$\pi_6((12)) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, we obtain that

$$\pi_6((123)) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$