1. Let W be the 2-dimensional irreducible module of S_3 where W acts on $\{a_1e_1+a_2e_2+a_3e_3 : a_1+a_2+a_3=0\} = < e_1-e_2, e_2-e_3 >$ whose codimension is 1 and V the standard (3-dimensional) one. There are two other irreducible representations of S_3 (since S_3 has 3 conjugacy classes there are a total of 3 irreducible representations as well): I (the trivial) and A (the alternating) where I acts on the S_3-stable subspace $\{a_1e_1+a_2e_2+a_3e_3 : a_1=a_2=a_3\}$ of V whose dimension is 1 and as for the representation A, it takes an element $\sigma \in S_3$ to +1 if σ is an even permutation and -1 if σ is an odd permutation. Thus the dimension of A is 1. Find the decomposition coefficients into irreducibles of for the following tensor products:

$$V \otimes V \tag{1}$$
$$V \wedge V \tag{2}$$
$$S^2V \tag{3}$$
$$W \otimes W \tag{4}$$

<u>Solution</u>.

For $V \otimes V \supseteq V \wedge V$: Recall that $V \cong I \oplus W$ where dim I=1 and dim W=2. Then

$$V \otimes V \cong (I \oplus W) \otimes (I \oplus W) \tag{5}$$
$$\cong (I \otimes I) \oplus (W \otimes I) \oplus (I \otimes W) \oplus (W \otimes W) \tag{6}$$
$$\cong I \oplus 2(W \otimes I) \oplus (W \otimes W) \tag{7}$$
$$\cong I \oplus 2W \oplus (W \oplus A \oplus I) \tag{8}$$
$$\cong 2I \oplus 3W \oplus A. \tag{9}$$

For $V \wedge V$: $V \wedge V = A \oplus W$ where $W =< e_1 - e_2, e_2 - e_3 >$.

For S^2V: In general if e_1,..., e_n is a basis of V, then $S^nV = \{e_{i_1} \cdot e_{i_2} \cdot ... \cdot e_{i_n} : i_1 \leq i_2 \leq ... \leq i_n\} \cong V \otimes V / < v_i \otimes v_j - v_j \otimes v_i >$. So for this particular problem $S^2V = \{e_1 \cdot e_2, e_1 \cdot e_3, e_2 \cdot e_3\} \oplus \{e_1 \cdot e_1, e_2 \cdot e_2, e_3 \cdot e_3\} = 2W \oplus 2I$.

For $W \otimes W$: From Tuesday's lecture, we saw that $W \otimes W \cong W \oplus A \oplus I$.