

HIGHER DIRECT IMAGES OF DUALIZING SHEAVES OF LAGRANGIAN FIBRATIONS

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ABSTRACT. We prove that the higher direct images of the dualizing sheaf of a Lagrangian fibration between smooth projective manifolds are isomorphic to the cotangent bundles of base space. As a corollary, we obtain that every Hodge number of the base space of a fibre space of an irreducible symplectic manifold is the same to that of a projective space if the base space is smooth.

1. INTRODUCTION

We begin with the definition of *Lagrangian fibrations*.

DEFINITION 1.1. *Let X be a Kähler manifold with a holomorphic symplectic form ω and S a normal variety. A proper surjective morphism $f : X \rightarrow S$ is said to be a Lagrangian fibration if a general fibre F of f is a Lagrangian submanifold with respect to ω , that is, the restriction of 2-form $\omega|_F$ is identically zero and $\dim F = (1/2) \dim X$.*

In this note, we investigate higher direct images of the dualizing sheaf of a Lagrangian fibration. Our result is the following.

THEOREM 1.2. *Let $f : X \rightarrow S$ be a Lagrangian fibration between smooth projective manifolds. Then*

$$R^i f_* \mathcal{O}_X \cong \Omega_S^i.$$

In particular, $R^i f_ \mathcal{O}_X$ are locally free.*

REMARK 1. It is known that the higher direct images of the dualizing sheaf are locally free if the discriminant locus is a normal crossing divisor [11, Theorem 2.6], [15, Theorem 2.6], and [16, Theorem 1]. However, there is a projective Lagrangian fibration whose discriminant locus is not a normal crossing divisor. The above theorem asserts that the higher direct images are locally free even if the discriminant locus is not necessarily normal crossing.

EXAMPLE. Let $X := \text{Hilb}^2 T$, where T is a projective $K3$ surface which has an elliptic fibration $T \rightarrow \mathbb{P}^1$. Then X admits a Lagrangian fibration over \mathbb{P}^2 and whose discriminant locus is not a normal crossing divisor.

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REMARK 2. The assumption projectivity is crucial in our present proof. The author suspect, however, that the most part of our results holds true for Lagrangian fibrations. Please see REMARK 4 in Section 3.

Combining Theorem 1.2 with [12, Theorem 2] and [13, Theorem 1], we obtain the following corollary.

COROLLARY 1.3. *Let $f : X \rightarrow S$ be a fibre space of a projective irreducible symplectic manifold over a smooth projective base S . Then S is Fano and hodge numbers of S are the following:*

$$h^q(S, \Omega_S^p) = \begin{cases} 1 & (p = q) \\ 0 & (p \neq q). \end{cases}$$

Every Hodge number of S is the same as that of \mathbb{P}^n , where $n = \dim S$.

REMARK 3. Every Hodge number of a quadric hypersurface Q in \mathbb{P}^{2n} is same to that of \mathbb{P}^{2n-1} and Q is not isomorphic to \mathbb{P}^{2n-1} ($n \geq 2$). The author does not know whether there exists an example such that a fibre space of an irreducible symplectic manifold $f : X \rightarrow S$ whose base space S is not isomorphic to a projective space. Recently, Miyaoka [2] announced that S is isomorphic to a projective space if f admits a local section. Note that if f admits a local section, then S is smooth. The author would like to state one question.

QUESTION 1.4. *Under the assumptions of Theorem 1.2, we have*

$$R^{n-1}f_*\omega_{X/S} \cong T_S,$$

which is the dual of $R^1f_\mathcal{O} \cong \Omega_S^1$. Note that $\omega_{X/S}$ is the relative dualizing sheaf and $n = \dim S$. Then $R^{n-1}f_*\omega_{X/S}$ is nef? In particular, $R^{n-1}f_*\omega_{X/S}$ is ample if X is an irreducible symplectic manifold?*

Note that it is known that $R^if_*\omega_{X/S}$ is weakly positive.

COROLLARY 1.5. *If $\dim X = 4$, then T_S is nef. In particular, T_S is ample if X is an irreducible symplectic manifold.*

This paper is organized as follows. In section 2, we prepare for the proof of Theorem 1.2. Theorem 1.2, Corollary 1.3 and Corollary 1.5 are proved in section 3. Section 4 is devoted to the classification of singular fibres over codimension one points which needs for the proof of Theorem 1.2.

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2. PRELIMINARY

We correct definitions and fundamental material which are needed in the proof of Theorem 1.2. First we consider a relative compactification of an Abelian fibration.

PROPOSITION 2.1. *Let $f^\circ : X^\circ \rightarrow \Delta^*(t_1) \times \Delta^{n-1}(t_2, \dots, t_n)$ be a smooth projective Abelian fibration. Assume that f° has the following properties:*

- (1) f° has a projective relative compactification $f' : Y \rightarrow \Delta^n$.
- (2) $f'^{-1}(p)$ has a reduced component for a general point p of the discriminant locus of f' .
- (3) The monodromy matrix T of $R^1 f'_* \mathbb{C}$ around $\Delta^*(t_1)$ is unipotent.
- (4) The rank of $T - I$ is at most one, where I is the identity matrix.

Then there exists a projective relative compactification $f : X \rightarrow \Delta^n$ which has the following properties:

- (1) Every singular fibre of f is an Abelian variety or a cycle of several copies of a \mathbb{P}^1 -bundle over an Abelian variety. In later case, $E_i \cap E_j$ forms a section of the ruling of E_i and E_j , where E_i and E_j are irreducible components of a singular fibre.
- (2) The morphism $f^{-1}(D) \rightarrow D$ is a locally trivial deformation, where D is the discriminant locus of f .
- (3) K_X is f -trivial.
- (4) Every fibre of f is reduced.

PROOF. This proposition is essentially deduced from [17, Theorem 5.3]. However, for a complete sake and the proof of projectivity of the relative compactification is not published, we give a proof. Let $\tau(t_1, \mathbf{t})$ be the period matrix of f° where $\mathbf{t} = (t_2, \dots, t_n)$, m is the dimension of a fibre of f° , I_k is the identity matrix of rank k and δ is the polarization matrix of f . We consider T as an element of $\mathrm{Sp}(m, \mathbb{R})$ by the conjugation

$$T' := \begin{pmatrix} I_m & 0 \\ 0 & \delta^{-1} \end{pmatrix} T \begin{pmatrix} I_m & 0 \\ 0 & \delta \end{pmatrix}.$$

Let G be the semiproduct of \mathbb{Z} and $\chi := \{a + b\delta; a, b \in \mathbb{Z}^{\oplus m}\}$ with commuting relations $(a + b\delta) \cdot \gamma = \gamma \cdot (a + b\delta) T'$ for $\gamma \in \mathbb{Z}$ and $a + b\delta \in \chi$. By the assumption, f' admits a meromorphic section locally over an open set V such that $\mathrm{codim}(\Delta^n \setminus V) \geq 2$. Since $\pi_1(V) \cong \pi_1(\Delta^n) = \{1\}$, f° admits a global section. Hence the original X°

is reconstructed as follows:

$$X^\circ = (\mathbb{H} \times \Delta^{n-1} \times \mathbb{C}^m)/G$$

where \mathbb{H} is an upper half space. The action of G is defined by

$$\begin{aligned} \gamma : \quad \mathbb{H} \times \Delta^{n-1} \times \mathbb{C}^m &\rightarrow \mathbb{H} \times \Delta^{n-1} \times \mathbb{C}^m \\ (s, \mathbf{t}, \mathbf{z}) &\mapsto (s+1, t, \mathbf{z}(C\tau(\exp(2\pi is), \mathbf{t}) + D)^{-1}) \\ a + b\delta : \mathbb{H} \times \Delta^{n-1} \times \mathbb{C}^m &\rightarrow \mathbb{H} \times \Delta^{n-1} \times \mathbb{C}^m \\ (s, \mathbf{t}, \mathbf{z}) &\mapsto (s, t, \mathbf{z} + (a + b\delta) \begin{pmatrix} \tau(\exp(2\pi is), \mathbf{t}) \\ I_m \end{pmatrix}), \end{aligned}$$

where

$$T' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

LEMMA 2.2. *If the rank of $T - I_{2m}$ is zero, then there exists a smooth Abelian fibration $f : X \rightarrow \Delta^n$ which is a relative compactification of X° . This fibration has properties of Proposition 2.1 (1), (2), (3) and (4).*

PROOF. By the assumption, the monodromy matrix $T = I_{2m}$. Therefore the period τ can be extended holomorphically over Δ^n . We construct X as follows:

$$X = (\Delta^n \times \mathbb{C}^m)/\chi,$$

where $\chi := \{a + b\delta; a, b \in \mathbb{Z}^{\oplus 2m}\}$ and the action of χ is

$$\begin{aligned} a + b\delta : \Delta^n \times \mathbb{C}^m &\rightarrow \Delta^n \times \mathbb{C}^m \\ (t_1, \mathbf{t}, \mathbf{z}) &\mapsto (t_1, \mathbf{t}, \mathbf{z} + (a + b\delta) \begin{pmatrix} \tau(t_1, \mathbf{t}) \\ I_m \end{pmatrix}). \end{aligned}$$

Since $f : X \rightarrow \Delta^n$ is a smooth morphism, it is obvious that X satisfies the assertions of Lemma. \square

We consider the case that the rank of $T - I_{2m}$ is one. Since the discriminant locus of g is smooth, $\lim_{t_1 \rightarrow 0} \tau(t_1, \mathbf{t})$ exists in a rational boundary component F_α of Siegel upper half plane and F_α does not depend \mathbf{t} . ([17, Remark (4.4)]). There exists a matrix $M \in \mathrm{Sp}(m, \mathbb{Z})$ which transform F_α to $F_{g'}$ (cf. [17, (2.3)]), where $F_{g'}$ is the rational boundary component of Siegel upper half plane defined by

$$F_{g'} := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & Z'' \end{pmatrix}; Z'' \in \mathrm{GL}(m - g', \mathbb{C}), I_{m-g'} - Z''\bar{Z}'' > 0 \right\}.$$

Since T' is unipotent, the transformation of T' by M can be written

$$(1) \quad MT'M^{-1} = \begin{pmatrix} I_m & B \\ 0 & I_m \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix},$$

by [17, (2.6)]. Note that $b \in \mathbb{R}^1$ because the rank of $T - I_{2m}$ is one. From this representation of the monodromy matrix $MT'M^{-1}$, the period matrix can be written as

$$M\tau(t_1, \mathbf{t}) = \begin{pmatrix} \tau'(t_1, \mathbf{t}) & \tau'''(t_1, \mathbf{t}) \\ {}^t\tau'''(t_1, \mathbf{t}) & \tau''(t_1, \mathbf{t}) \end{pmatrix} + \frac{1}{2\pi i} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \log t_1,$$

where each $\tau^*(t_1, \mathbf{t})$ be a holomorphic function on Δ^n . From [17, (2.9)], $\text{Im}\tau' > 0$ and $\text{Im}\tau'' > 0$. We denote each element of χM^{-1} as

$$\{(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \chi M^{-1}; \alpha_1, \beta_1 \in \mathbb{R}^{m-1}, \alpha_2, \beta_2 \in \mathbb{R}^1\}.$$

Let $\bar{\chi}$ be the sublattice of χM^{-1} such that

$$\bar{\chi} := \{(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \chi M^{-1}; \alpha_1 = \alpha_2 = \beta_1 = 0\}.$$

and $(0, 0, 0, c)$ its generator. We define a toric variety \mathcal{M} for the construction X .

DEFINITION 2.3. *Let σ_k be a cone defined by*

$$\sigma_k := \{(u, y) \in \mathbb{R}^2; ku \leq y \leq (k+1)u\}$$

Then $\mathbb{C}[\sigma_k^\vee \cap \mathbb{Z}^2]$ is isomorphic to \mathbb{C} -subalgebra of $\mathbb{C}[v^{\pm 1}, t_1^{\pm 1}]$ generated by monomials $\{t_1^{-k}v, t_1^{k+1}v^{-1}\}$ and there is a natural morphism $\text{Spec}\mathbb{C}[\sigma_k^\vee \cap \mathbb{Z}^2] \rightarrow \text{Spec}\mathbb{C}[t_1]$. We define the toric variety \mathcal{M} as follows:

$$\mathcal{M} := \bigcup_{k \in \mathbb{Z}} (\text{Spec}\mathbb{C}[\sigma_k^\vee \cap \mathbb{Z}^2]) \times_{\mathbb{C}^1} \Delta^1.$$

Note that \mathcal{M} has a fibration $\mathcal{M} \rightarrow \Delta^1$ and whose singular fibre is an infinite chain of \mathbb{P}^1 .

LEMMA 2.4. *Let $\mathcal{M}^* := \mathcal{M} \times_{\Delta^1} \Delta^*$. Then $\mathcal{M}^* \cong \Delta^* \times \mathbb{C}^*$ and the variable v is considered to be a coordinate of \mathbb{C}^* . We define the action of χM^{-1} on $\Delta^{n-1} \times \mathbb{C}^{m-1} \times \mathcal{M}^*$ as follows:*

$$\Gamma_{(\alpha_1, \alpha_2, \beta_1, \beta_2)} : (\mathbf{t}, t_1, \mathbf{z}, v) \mapsto (\mathbf{t}, t_1, \mathbf{z} + A, Bv),$$

where

$$\begin{aligned} A &= \alpha_1 \tau'(t_1, \mathbf{t}) + \beta_1 + \alpha_2 \tau'''(t_1, \mathbf{t}) \\ B &= \exp\left(\frac{2\pi i}{c}(\alpha_1 {}^t\tau'''(t_1, \mathbf{t}) + \alpha_2 \tau''(t_1, \mathbf{t}) + \beta_2)\right) t_1^{\alpha_2 b/c}. \end{aligned}$$

Then the quotient $(\Delta^{n-1} \times \mathcal{M}^ \times \mathbb{C}^{m-1})/\chi M^{-1} \cong X^\circ$.*

PROOF. We begin with to prove that the action is well defined. It is enough to show that $\alpha_2 b/c$ is an integer for every element of χM^{-1} . For each element x of χ , $xT' \in \chi$. Hence $\chi(T' - I_{2m}) \subset \chi$ and we have $\chi M^{-1}(MT'M^{-1} - I_{2m}) \subset \chi M^{-1}$. On the contrary, $(0, 0, 0, \alpha_2 b) = (\alpha_1, \alpha_2, \beta_1, \beta_2)(MT'M^{-1} - I_{2m})$. Therefore

$(0, 0, 0, \alpha_2 b) \in \bar{\chi}$ for every element of χM^{-1} and $\alpha_2 b/c$ is an integer. If we consider the morphism

$$\begin{aligned} \Delta^{n-1} \times \Delta^* \times \mathbb{C}^{m-1} \times \mathbb{C} &\rightarrow \Delta^{n-1} \times \Delta^* \times \mathbb{C}^{m-1} \times \mathbb{C}^* \\ (\mathbf{t}, t_1, \mathbf{z}, z) &\mapsto (\mathbf{t}, t_1, \mathbf{z}, \exp(2\pi iz/c)), \end{aligned}$$

and rewrite the construction of X° by $MT'M^{-1}$ and $M\tau$, we obtain that $\Delta^{n-1} \times \mathcal{M}^* \times \mathbb{C}^{m-1} / \chi M^{-1} \cong X^\circ$. \square

LEMMA 2.5. *The action of χM^{-1} in Lemma 2.4 can be extended on $\Delta^{n-1} \times \mathbb{C}^{n-1} \times \mathcal{M}$. This action is properly discontinuous and fixed points free. The quotient X is a relative compactification of X° and $f : X \rightarrow \Delta^n$ has properties of Proposition 2.1 (1), (2), (3) and (4).*

PROOF. By definition, every singular fibre of $\Delta^{n-1} \times \mathcal{M} \times \mathbb{C}^{n-1} \rightarrow \Delta^n$ is the product of an infinite chain of \mathbb{P}^1 and \mathbb{C}^{n-1} . We denote each irreducible component of a singular fibre by V_k . Since the action of χM^{-1} on \mathcal{M} maps $\text{Spec}\mathbb{C}[\sigma_k \cap \mathbb{Z}^2]$ to $\text{Spec}\mathbb{C}[\sigma_{k+(\alpha_2 b)/c} \cap \mathbb{Z}^2]$, V_k maps to $V_{k+\alpha_2 b/c}$ by this action. Hence this action is properly discontinuous and fixed point free. The quotient X is smooth and a relative compactification of X° . By construction, it is obvious that f satisfies the properties (1) and (2) of Proposition 2.1. We will prove that every fibre of f is reduced. By construction, the multiplicity of each component of a singular fibre is same. Thus it is enough to prove that every fibre has a reduced component. The section $\Delta^n \ni (\mathbf{t}, t_1) \mapsto (\mathbf{t}, t_1, 1, 0) \in \Delta^{n-1} \times \mathcal{M} \times \mathbb{C}^{n-1}$ induces a global section of f . Hence every fibre of f is reduced. Moreover, K_X is f -trivial because there is the holomorphic $n + m$ -form

$$dt_1 \wedge \cdots \wedge dt_n \wedge dz_1 \wedge \cdots \wedge dz_{m-1} \wedge \frac{dv}{v}$$

which is χM^{-1} -invariant and nowhere vanishing on $\Delta^{n-1} \times \mathcal{M} \times \mathbb{C}^{m-1}$. \square

The rest of assertions of Proposition 2.1 is deduced from the following Lemma.

LEMMA 2.6. *Let $f : X \rightarrow \Delta^n$ be the relative compactification constructed in Lemma 2.2 or Lemma 2.5. Then f is projective.*

PROOF. By changing birational model, we may assume that there is a morphism $Y \rightarrow X$. Let H be a relative ample divisor on Y and H' its proper transform on X . Then H' is a big divisor on each component of a fibre of f . If $T = I_{2m}$, every fibre of f is an Abelian variety. Thus every big divisor on each component of a fibre of f is ample. Therefore H' is ample. If $T \neq I_{2m}$, we need the following claim:

CLAIM 2.7. *Let $f : X \rightarrow \Delta^n$ be the relative compactification constructed in Lemma 2.5 and E_s an irreducible component of a singular fibre of f at s . Then there is a section e of the ruling of E_s such that $-K_{E_s} \equiv 2e$ and e is nef.*

PROOF. Let C be a smooth curve on Δ^n such that $C \cap D = \{s\}$, where D is the discriminant locus of f . We consider the restriction $f_C : X_C := X \times_{\Delta^n} C \rightarrow C$. We denote each irreducible component of the singular fibre of f_C by $E_{i,s}$. Let $E_{i+1,s}$ and $E_{i-1,s}$ be next components of $E_{i,s}$ and $F_{ij,s} := E_{i,s} \cap E_{j,s}$. Since K_{X_C} is f_C -trivial, $K_{E_{i,s}} \sim -F_{ii+1,s} - F_{ii-1,s}$. Each $E_{i,s}$ is a \mathbb{P}^1 -bundle over an Abelian variety. Hence $-K_{E_{i,s}}$ is nef. In order to prove Claim, it is enough to show that $F_{ii+1,s}$ and $F_{ii-1,s}$ are numerically equivalent on $E_{i,s}$. Since X_s is a deformation retract of X_C , there is a collapsing morphism $c_t : X_t \rightarrow X_s$, where X_t is a general fibre of f_C . Let $E'_{i,s} := c_t^{-1}(E_{i,s})$ and $F'_{ij,s} := c_t^{-1}(F_{ij,s})$. Note that $E'_{i,s} \rightarrow E_{i,s}$ is a real blow up along $F_{ij,s}$ and $F'_{ij,s}$ is a S^1 -bundle over $F_{ij,s}$. We consider the following Mayer-Vietoris sequences:

$$\begin{aligned} 0 \rightarrow \mathbb{C}_{X_t} &\rightarrow \bigoplus_{E'_{i,s}} \mathbb{C}_{E'_{i,s}} \rightarrow \bigoplus_{F'_{ij,s}} \mathbb{C}_{F'_{ij,s}} \rightarrow 0 \\ 0 \rightarrow \mathbb{C}_{X_s} &\rightarrow \bigoplus_{E_{i,s}} \mathbb{C}_{E_{i,s}} \rightarrow \bigoplus_{F_{ij,s}} \mathbb{C}_{F_{ij,s}} \rightarrow 0 \end{aligned}$$

From the above sequence,

$$0 \rightarrow H^0(X_t) \rightarrow \bigoplus_{E'_{i,s}} H^0(E'_{i,s}) \rightarrow \bigoplus_{F'_{ij,s}} H^0(F'_{ij,s}) \xrightarrow{\alpha_1} H^1(X_t),$$

and

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ \rightarrow H^1(X_s) & \rightarrow & \bigoplus_{E_{i,s}} H^1(E_{i,s}) & \xrightarrow{\alpha_4} & \bigoplus_{F_{ij,s}} H^1(F_{ij,s}) & \rightarrow & \\ & & \downarrow & & \downarrow & & \\ \xrightarrow{\alpha_1} H^1(X_t) & \xrightarrow{\alpha_2} & \bigoplus_{E'_{i,s}} H^1(E'_{i,s}) & \xrightarrow{\alpha_3} & \bigoplus_{F'_{ij,s}} H^1(F'_{ij,s}) & \rightarrow & H^2(X_t) \rightarrow \\ & & \downarrow & & \downarrow & & \\ 0 \rightarrow \bigoplus_{F_{ij,s}} H^0(F_{ij,s}) & \rightarrow & \bigoplus_{E_{i,s} \supset F_{ij,s}} H^0(F_{ij,s}) & \rightarrow & \bigoplus_{F_{ij,s}} H^0(F_{ij,s}) & \rightarrow & 0 \\ & & \downarrow \beta & & \downarrow & & \\ & & \bigoplus_{E_{i,s}} H^2(E_{i,s}) & \rightarrow & \bigoplus_{F_{ij,s}} H^2(F_{ij,s}) & & \end{array}$$

Note that each column is Gysin sequence. Then $\dim \text{Im}(\alpha_1) = 1$ and $\dim \text{Ker}(\alpha_4) = 2n - 2$, because the singular fibre of f_C satisfies the property (1) of Proposition 2.1. Since $\dim \text{Im}(\alpha_1) + \dim \text{Im}(\alpha_2) = 2n$, $\dim \text{Ker}(\alpha_3) = 2n - 1$. Hence β is not isomorphism. The morphism β sends

$$H^0(F_{ii-1,s}) \oplus H^0(F_{ii+1,s}) \ni (\lambda_1, \lambda_2) \mapsto (\lambda_1[F_{ii-1,s}] + \lambda_2[F_{ii+1,s}]) \in H^2(E_{i,s}).$$

If $F_{ii-1,s}$ and $F_{ii+1,s}$ is not numerically equivalent, β is injection. That derives a contradiction and we are done. \square

We go back to the proof of Lemma. From Claim 2.7, the nef cone of each irreducible component of a singular fibre is generated by a section and divisors on an Abelian variety. Therefore every big divisor on each irreducible component of singular fibre is ample and projectivity of f can be shown similarly as in the case $T = I_{2m}$. \square

We complete the proof of Proposition 2.1. \square

For the using later, we prove the following Lemma.

LEMMA 2.8. *Let $f : X \rightarrow \Delta^n$ be a projective Abelian fibration which satisfies the properties (1), (2) and (3) of Proposition 2.1. Then every birational map $\Phi : X \dashrightarrow X$ which commutes with f can be extended to a biholomorphic morphism, that is, X is the unique relative minimal model.*

PROOF . We derive a contradiction assuming that Φ is not isomorphism. Since X is a relative minimal model and Φ commutes with f , Φ is an isomorphism over codimension one point of X . Let H be an ample divisor on X and H' the proper transform by Φ . If Φ is not isomorphism, then H' is not f -nef. Hence by [6, Theorem 3-1-1], there exists an extremal contraction $\nu : X \rightarrow \bar{X}$. Let ℓ be a rational curve which is contracted by ν and E an irreducible component of $f^{-1}(D)$ which contains ℓ . Since f satisfies the properties (1) and (2) of Proposition 2.1 and the discriminant locus D of f is simply connected, $E \rightarrow D$ factors $E \rightarrow A \rightarrow D$, where $E \rightarrow A$ is a \mathbb{P}^1 -bundle and $A \rightarrow D$ is a smooth Abelian fibration. Hence ℓ must be a fibre of the ruling of E . Thus ν contracts every fibre of the ruling of E and the exceptional locus of ν has codimension one. That derives a contradiction. \square

DEFINITION 2.9. *Let $f : X \rightarrow \Delta^n$ be a projective Abelian fibration which satisfies the properties of Proposition 2.1 (1) - (4). We call X as a toroidal model of type I or II according to the singular fibre of f is an Abelian variety or a cycle of several copies of a \mathbb{P}^1 -bundle over an Abelian variety.*

Next we note variations of Hodge structures. Let $f : X \rightarrow S$ be a projective morphism between smooth manifold. Assume that there exists an open set U of S such that f is smooth over U and $D := S \setminus U$ is a simple normal crossing divisor. Let $f^\circ : f^{-1}(U) \rightarrow U$ be the smooth part of f . We denote the upper canonical extension of $\mathcal{H} := R^i f_* \mathbb{C}$ by ${}^u\mathcal{H}$ and the lower canonical extension by ${}^\ell\mathcal{H}$. For the proof of Theorem 1.2, we need the following Lemma.

LEMMA 2.10. *Let $f : X \rightarrow \Delta^n$ be a projective morphism from a smooth manifold X to a polydisk Δ^n . Assume that f has the following properties:*

- (1) *The discriminant locus D of f is smooth and $E := (f^*D)_{\text{red}}$ is a simple normal crossing divisor.*
- (2) *The morphism $E \rightarrow D$ is a locally trivial deformation.*

Let $f^*D = \sum a_i E_i$. Then

$${}^u\mathcal{F}^1(\mathcal{H}) \cong f_*(\Omega_{X/\Delta^n}^1(\log E) \otimes \sum (a_i - 1)E_i).$$

PROOF. If $\dim S = 1$, then this assertion is [5, VIII]. Thus ${}^u\mathcal{F}^1(\mathcal{H})$ and $f_*(\Omega_{X/\Delta^n}^1(\log E) \otimes \sum (a_i - 1)E_i)$ are isomorphic in codimension one points of S . By the assumptions of f , [18, Theorem 3.5] and the argument in the proof of [11, Lemma 2.11], $f_*(\Omega_{X/\Delta^n}^1(\log E) \otimes \sum (a_i - 1)E_i)$ is locally free. Therefore we obtain the assertion of Lemma 2.10. \square

3. CORRESPONDENCE BETWEEN HIGHER DIRECT IMAGES AND COTANGENT BUNDLES

In this section, we prove Theorem 1.2, Corollary 1.3 and Corollary 1.5. We begin with to consider the relation among $R^i f_* \mathcal{O}_X$. From the following lemma, it is enough to consider $R^1 f_* \mathcal{O}_X$ for the proof of Theorem 1.2.

LEMMA 3.1. *Let $f : X \rightarrow S$ be a Lagrangian fibration between smooth projective manifolds. Assume that $R^1 f_* \mathcal{O}_X \cong \Omega_S^1$. Then*

$$R^i f_* \mathcal{O}_X \cong \Omega_S^i.$$

PROOF. Since $\omega_X \cong \mathcal{O}_X$, $R^i f_* \mathcal{O}_X$ are torsion free by [10, Theorem 2.1]. Thus $R^i f_* \mathcal{O}_X$ are reflexive sheaves by [11, Corollary 3.9]. Therefore it is enough to show that there exists an open set U of S such that $R^i f_* \mathcal{O}_X|_U \cong \Omega_U^i$ and $\text{codim}(S \setminus U) \geq 2$. Let D be the discriminant locus of f . We choose an open set U of S such that D is smooth in U . Since a general fibre of f is an Abelian variety (Liouville's Theorem) there exists a morphism

$$\wedge^i R^1 f_* \mathcal{O}_X|_U \xrightarrow{\alpha_i} R^i f_* \mathcal{O}_X|_U.$$

Then α is injective because $R^1 f_* \mathcal{O}_X|_U$ is torsion free. By [11, Theorem 2.6], $R^i f_* \mathcal{O}_X|_U \cong {}^\ell\mathcal{G}r^i(\mathcal{H})$. Considering exponents of the lower extension of \mathcal{H} , α_n is not surjective if α_i ($0 < i < n$) is not surjective. By [10, Corollary 7.6], $R^n f_* \mathcal{O}_X \cong \omega_S$. Combining $R^1 f_* \mathcal{O}_X \cong \Omega_S^1$, α_n is isomorphism. Hence α_i is isomorphism for $1 \leq i \leq n$. \square

REMARK 4. The reflexiveness of $R^i f_* \mathcal{O}_X$ is one of crucial point why we need projectivity. This comes from the following decomposition theorem.

THEOREM 3.2 ([11, Theorem 3.1]). *Let $f : X \rightarrow S$ be a surjective morphism between projective variety. If X is smooth, then*

$$Rf_* \omega_X \sim_{\text{q.i.s.}} \sum R^i f_* \omega_X[-i].$$

If the above theorem valid for a projective morphism, we can replace projectivity of X by projectivity of f in the assumptions of Theorem 1.2.

Theorem 1.2 can be deduced from the following proposition.

PROPOSITION 3.3. *Let $f : X \rightarrow S$ be a projective Lagrangian fibration with a symplectic form ω . Then there exists an open set U of S such that*

$$R^1 f_* \mathcal{O}_X|_U \cong \Omega_U^1$$

and $\text{codim}(S \setminus U) \geq 2$.

PROOF. We prove Proposition 3.3 in four steps.

STEP 1. First we consider the smooth part of f . Let $U_0 := S \setminus D$, where D is the discriminant locus of f . We consider the following exact sequence:

$$\begin{array}{ccccccc} & & T_{f^{-1}(U_0)} & \rightarrow & f^* T_{U_0} & \rightarrow & 0 \\ & & \downarrow \omega & & & & \\ 0 & \rightarrow & f^* \Omega_{U_0}^1 & \rightarrow & \Omega_{f^{-1}(U_0)}^1 & \rightarrow & \Omega_{f^{-1}(U_0)/U_0}^1 \rightarrow 0 \end{array}.$$

From the above diagram, ω defines a morphism

$$f^* T_{U_0} \rightarrow \Omega_{f^{-1}(U_0)/U_0}^1$$

because every fibre of f is a Lagrangian subvariety by [14, Theorem 1]. Since ω is nondegenerate, the above morphism is isomorphism. Therefore we obtain isomorphisms

$$T_{U_0} \cong f_* \Omega_{f^{-1}(U_0)/U_0}^1 \cong \mathcal{F}^1(\mathcal{H}).$$

Taking the dual of above isomorphism,

$$(2) \quad R^1 f_* \mathcal{O}_X|_{U_0} \cong \Omega_{U_0}^1.$$

STEP 2. In order to prove Proposition 3.3, we extend the isomorphism (2) over codimension one point of S . Let s be a point of D such that S and D are smooth around s . In Step 2, 3 and 4, we will show that the isomorphism (2) can be extended on a neighborhood of s . In step 2, we treat the case that $f^{-1}(s)$ satisfies the properties of Theorem 4.1 (1) and $G_1 = G_2 = \{1\}$. In this case, there exists a polydisk $s \in \Delta^n$ of S and the restriction morphism $f : X_{\Delta^n} := X \times_S \Delta^n \rightarrow \Delta^n$ is a toroidal model of type I or II (See Definition 2.9). If f is a toroidal model of type I, then f is smooth and we obtain $R^1 f_* \mathcal{O}_{X_{\Delta^n}} \cong \Omega_{\Delta^n}^1$ around s . Thus we may assume that f is a toroidal model of type II. Let

$$X^\circ := \{x \in X_{\Delta^n} | f \text{ is smooth at } x\}.$$

We consider the following diagram:

$$\begin{array}{ccccccc} T_{X^\circ} & \rightarrow & f^*T_{\Delta^n}|_{X^\circ} & \rightarrow & 0 & & \\ & & \downarrow \omega & & & & \\ 0 & \rightarrow & f^*\Omega_{\Delta^n}^1|_{X^\circ} & \rightarrow & \Omega_{X^\circ}^1 & \rightarrow & \Omega_{X^\circ/\Delta^n}^1 \rightarrow 0 \end{array}.$$

From the above diagram, ω defines a morphism $f^*T_{\Delta^n}|_{X^\circ} \rightarrow \Omega_{X^\circ/\Delta^n}^1$ since every fibre of f is a Lagrangian subvariety by [14, Theorem 1]. We investigate the relation between $\Omega_{X^\circ/\Delta^n}^1$ and $\Omega_{X_{\Delta^n}/\Delta^n}^1(\log E)|_{X^\circ}$. Let us consider the following diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & f^*\Omega_{\Delta^n}^1|_{X^\circ} & \rightarrow & \Omega_{X^\circ}^1 & \rightarrow & \Omega_{X^\circ/\Delta^n}^1 \rightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \rightarrow & f^*\Omega_{\Delta^n}^1(\log D)|_{X^\circ} & \rightarrow & \Omega_{X_{\Delta^n}}^1(\log E)|_{X^\circ} & \rightarrow & \Omega_{X_{\Delta^n}/\Delta^n}^1(\log E)|_{X^\circ} \rightarrow 0. \end{array}$$

We obtain $\Omega_{X^\circ/S}^1 \cong \Omega_{X_{\Delta^n}/\Delta^n}^1(\log E)|_{X^\circ}$ if $\text{Coker}\alpha \cong \text{Coker}\beta$. Let x be a point of $f^{-1}(D)$. From Definition 2.9, if we choose a suitable local parameter z_i at x and t_i at $f(x)$, f is written as

$$z_0z_1 = t_1, \quad z_i = t_i \quad (2 \leq i),$$

on a suitable neighborhood of x . The generator of $\text{Coker}\alpha$ is f^*dt_1/t_1 and the generators of $\text{Coker}\beta$ are dz_0/z_0 and dz_1/z_1 . The morphism $\text{Coker}\alpha \rightarrow \text{Coker}\beta$ sends

$$f^*dt_1/t_1 \mapsto dz_0/z_0 + dz_1/dz_1.$$

If $x \in X^\circ$, $z_0 \neq 0$ or $z_1 \neq 0$. Hence $\text{Coker}\alpha \rightarrow \text{Coker}\beta$ is an isomorphism and we obtain $\Omega_{X^\circ/\Delta^n}^1 \cong \Omega_{X_{\Delta^n}/\Delta^n}^1(\log E)|_{X^\circ}$. We conclude that

$$f^*T_{\Delta^n} \cong \Omega_{X_{\Delta^n}/\Delta^n}^1(\log E),$$

because $\text{codim}(X \setminus X^\circ) \geq 2$. Since $f_*\Omega_{X_{\Delta^n}/\Delta^n}^1(\log E) \cong \mathcal{F}^1(\mathcal{H})$, we obtain

$$R^1f_*\mathcal{O}_{X_{\Delta^n}} \cong \Omega_{\Delta^n}^1.$$

STEP 3. In this step, we treat that $f^{-1}(s)$ satisfies the properties of Theorem 4.1 (1) and $G_1 \neq \{1\}$ or $G_2 \neq \{1\}$. By Theorem 4.1 (1), there exists a following diagram:

$$\begin{array}{ccccccc} X_{\Delta^n} & \xrightarrow{\eta} & \bar{X}/G_1 & \xleftarrow{\nu_1} & \bar{X} & = & \tilde{X}/G_2 \xleftarrow{\nu_2} \tilde{X} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Delta^n & = & \bar{\Delta}^n/G_1 & \leftarrow & \bar{\Delta}^n & = & \tilde{\Delta}^n/G_2 \leftarrow \tilde{\Delta}^n. \end{array}$$

First we consider the case $G_1 = \{1\}$ and $G_2 \neq \{1\}$. Since η is isomorphism on the smooth locus of \tilde{X}/G , $X_{\Delta^n} \cong \tilde{X}/G_2$. For a symplectic form ω on X , $\nu_2^*\omega$ is a symplectic form on \tilde{X} because ν_2 is étale. By Step 2, $\nu_2^*\omega$ defines an G_2 -equivariant isomorphism $R^1\tilde{f}_*\mathcal{O}_{\tilde{X}} \cong \Omega_{\Delta^n}^1$. Take the G_2 -invariant part, we obtain

$$R^1f_*\mathcal{O}_{X_{\Delta^n}} \cong \Omega_{\Delta^n}^1.$$

Next we consider the case $G_1 \neq \{1\}$. Let U_1 be the smooth locus of \bar{X}/G . Since η is isomorphism on U_1 and $\text{codim}((\bar{X}/G) \setminus U_1) \geq 2$, we define $\eta_*\omega$ as the extension of $\omega|_{U_1}$. On the contrary, ν_1 is étale in codimension one, $\nu_1^*(\eta_*\omega)$ is a symplectic form on \bar{X} . According to the argument in the case $G_1 = \{1\}$, $\nu_1^*(\eta_*\omega)$ defines a G_1 -equivariant isomorphism

$$R^1 f_* \mathcal{O}_{\bar{X}} \cong \Omega_{\bar{\Delta}^n}^1.$$

Take a G_1 -invariant part of the above morphism, we obtain

$$R^1 f_* \mathcal{O}_{X_{\Delta^n}} \cong \Omega_{\Delta^n}^1.$$

STEP 4. In this step, we treat that $f^{-1}(s)$ satisfies the properties of Theorem 4.1 (2). If we choose a suitable neighborhood Δ^n of s , there is an Abelian fibration $\tilde{f} : \tilde{X} \rightarrow \tilde{\Delta}^n$ and the restriction $f : X_{\Delta^n} := X \times_S \Delta^n \rightarrow \Delta^n$ which satisfy the following diagram:

$$\begin{array}{ccccc} X_{\Delta^n} & = & \tilde{X}/G & \xleftarrow{\tau} & \tilde{X} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^n & = & \tilde{\Delta}^n/G & \xleftarrow{} & \tilde{\Delta}^n, \end{array}$$

where τ is an étale morphism. Since τ is étale, $\tilde{\omega} := \tau^*\omega$ is a symplectic form where ω is a symplectic form on X_{Δ^n} . If $\tilde{\omega}$ defines an isomorphism $R^1 \tilde{f}_* \mathcal{O}_{\tilde{X}} \rightarrow \Omega_{\tilde{\Delta}^n}^1$ we obtain a $R^1 f_* \mathcal{O}_{X_{\Delta^n}} \cong \Omega_{\Delta^n}^1$ by taking a G -invariant part. Thus we consider the Abelian fibration $\tilde{f} : \tilde{X} \rightarrow \tilde{\Delta}^n$. By Theorem 4.1 (2), there is a birational morphism $\nu : W \rightarrow \tilde{X}$ such that $\tilde{f} \circ \nu : W \rightarrow \tilde{\Delta}^n$ satisfies the assumptions (1) and (2) of Lemma 2.10. Let $E := ((\tilde{f} \circ \nu)^* \tilde{D})_{red}$ and $(\tilde{f} \circ \nu)^* \tilde{D} = \sum a_i E_i$. We will show that $\nu^* \tilde{\omega}$ defines the morphism

$$(3) \quad (\tilde{f} \circ \nu)^* T_{\tilde{\Delta}^n} \rightarrow \Omega_{W/\tilde{\Delta}^n}^1(\log E) \otimes \sum (a_i - 1) E_i.$$

Let w be a point of $(\tilde{f} \circ \nu)^{-1}(\tilde{D})$. On a suitable neighborhood of w , $\tilde{f} \circ \nu$ is written as

$$z_0^{a_0} \cdots z_k^{a_k} = t_1, \quad z_{k+i} = t_i \quad (i \geq 2),$$

where z_i are local parameter at w and t_i are local parameter at $\tilde{f} \circ \nu(w)$. The morphism $T_W \rightarrow (\tilde{f} \circ \nu)^* T_{\tilde{\Delta}^n}$ is written as

$$\frac{\partial}{\partial z_j} \mapsto a_j z_0^{a_0} \cdots z_j^{a_j-1} \cdots z_k^{a_k} \frac{\partial}{\partial t_1}.$$

Let $W_0 := W \setminus \text{Sing}(\tilde{f} \circ \nu)^{-1}(\tilde{D})$. If $w \in W_0$, one of z_i ($0 \leq i \leq k$) is nonzero. Thus

$$(\tilde{f} \circ \nu)^* T_{\tilde{\Delta}^n} \otimes \sum (1 - a_i) E_i|_{W_0} \subset \text{Image}(T_W|_{W_0} \rightarrow (\tilde{f} \circ \nu)^* T_{\tilde{\Delta}^n}|_{W_0}).$$

We consider the following diagram:

$$\begin{array}{ccccccc}
0 \rightarrow & T_{W/\tilde{\Delta}^n} & \rightarrow & T_W & \rightarrow & (\tilde{f} \circ \nu)^* T_{\tilde{\Delta}^n} & \\
& & & \downarrow & & & \\
& (\tilde{f} \circ \nu)^* \Omega_{\tilde{\Delta}^n}^1 & \rightarrow & \Omega_W^1 & \rightarrow & \Omega_{W/\tilde{\Delta}^n}^1 & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & (\tilde{f} \circ \nu)^* \Omega_{\tilde{\Delta}^n}^1(\log \tilde{D}) & \rightarrow & \Omega_W^1(\log E) & \rightarrow & \Omega_{W/\tilde{\Delta}^n}^1(\log E) & \rightarrow 0.
\end{array}$$

By [14, Theorem 1], the restriction of $\nu^* \tilde{\omega}$ to every irreducible component of every fibre is identically zero. Thus $\nu^* \tilde{\omega}(\alpha, *) = 0$ in $\Omega_{W/\tilde{\Delta}^n}^1$ for all elements $\alpha \in T_{W/\tilde{\Delta}^n}$. Hence $\nu^* \tilde{\omega}$ defines the morphism

$$(\tilde{f} \circ \nu)^* T_{\tilde{\Delta}^n} \otimes \sum (1 - a_i) E_i|_{W_0} \rightarrow \Omega_{W/\tilde{\Delta}^n}^1(\log E)|_{W_0}.$$

Since $\text{codim}(W \setminus W_0) \geq 2$, we obtain the morphism (3). Let $\tilde{X}^\circ := \{\tilde{x} \in \tilde{X}; \tilde{f} \text{ is smooth at } \tilde{x}\}$ and V the open set of \tilde{X} such that $\nu^{-1}(V) \rightarrow V$ is an isomorphism. Note that $a_i = 1$ if $E_i \cap \nu^{-1}(X^\circ) \neq \emptyset$. By similar argument in Step 2, we obtain

$$\Omega_{W/\tilde{\Delta}^n}^1(\log E) \otimes \left(\sum (a_i - 1) E_i \right)|_{\nu^{-1}(\tilde{X}^\circ)} \cong \Omega_{\nu^{-1}(\tilde{X}^\circ)/\tilde{\Delta}^n}^1.$$

Since $\nu^* \tilde{\omega}$ is nondegenerate on $\nu^{-1}(V)$, the morphism (3) is an isomorphism on $\nu^{-1}(\tilde{X}^\circ \cap V)$. Thus we obtain an isomorphism

$$(4) \quad \tilde{f}^* T_{\tilde{\Delta}^n}|_{\tilde{X}^\circ \cap V} \rightarrow \nu_* \left(\Omega_{W/\tilde{\Delta}^n}^1(\log E) \otimes \sum (a_i - 1) E_i \right)|_{\tilde{X}^\circ \cap V}.$$

By Theorem 4.1 (2), every fibre of \tilde{f} is reduced. Hence $X^\circ = X \setminus \text{Sing}(\tilde{f}^{-1}(\tilde{D}))$ and $\text{codim}(\tilde{X} \setminus \tilde{X}^\circ) \geq 2$. Combining with $\text{codim}(\tilde{X} \setminus V) \geq 2$, $\text{codim}(\tilde{X} \setminus (\tilde{X}^\circ \cap V)) \geq 2$. Since $\tilde{f}^* T_{\tilde{\Delta}^n}$ is locally free, we obtain the morphism (4) is an isomorphism on \tilde{X} and

$$T_{\tilde{\Delta}^n} \cong \tilde{f}_* \left(\Omega_{W/\tilde{\Delta}^n}^1(\log E) \otimes \sum (a_i - 1) E_i \right).$$

By Lemma 2.10, $\tilde{f}_* \left(\Omega_{W/\tilde{\Delta}^n}^1(\log E) \otimes \sum (a_i - 1) E_i \right) \cong {}^u \mathcal{F}^1(\mathcal{H})$. We conclude that $T_{\tilde{\Delta}^n} \cong {}^u \mathcal{F}^1(\mathcal{H})$ and $R^1 \tilde{f}_* \mathcal{O}_{\tilde{X}} \cong \Omega_{\tilde{\Delta}^n}^1$. We complete the proof of Proposition 3.3. \square

PROOF OF THEOREM 1.2. Under the assumption of Theorem 1.2, $R^1 f_* \mathcal{O}_X$ is a reflexive sheaf by [11, Corollary 3.9]. Hence we obtain $R^1 f_* \mathcal{O}_X \cong \Omega_S^1$ by Proposition 3.3. Then by Lemma 3.1, we obtain the assertion of Theorem 1.2. \square

PROOF OF COROLLARY 1.3. By [12, Theorem 2] and [13, Theorem 1], $f : X \rightarrow S$ is a Lagrangian fibration and S is a Fano manifold. From [11, Corollary 3.2],

$$h^k(X, \omega_X) = \sum_{p+q=k} h^q(S, R^p f_* \omega_X).$$

Since $\omega_X \cong \mathcal{O}_X$, the right hand side of the above equation is the sum of Hodge numbers of S by Theorem 1.2. By Hodge conjugate $h^k(X, \mathcal{O}_X) = h^0(X, \Omega_X^k)$. Moreover

$$h^0(X, \Omega_X^k) = \begin{cases} 0 & k \equiv 1 \pmod{2} \\ 1 & k \equiv 0 \pmod{2} \end{cases}$$

since X is an irreducible symplectic manifold. From the assumption that S is projective, $h^{p,p}(S) \geq 1$. Combining the above results, we obtain the assertion of Corollary 1.3. \square

PROOF OF COROLLARY 1.5. We classify S by using the classification of algebraic surfaces. We begin with to prove that S does not contain curves whose selfintersection number are negative. Assume the contrary. Then there is a birational morphism $\pi : S \rightarrow \bar{S}$ which contracts curves to points. The composition morphism $\pi \circ f$ is a Lagrangian fibration and not equidimensional. This contradicts to [14, Theorem 1]. Next we investigate the irregularity and Kodaira dimension of S . From [8, Theorem 1.1], $\kappa(S) \leq 0$. By similar argument in the proof of Corollary 1.3, $2q(S) = q(X)$. Since $\kappa(X) = 0$, $q(X) \leq 4$ and $q(S) \leq 2$. If $q(S) = 2$, X is an Abelian variety. From the following Claim, S is an Abelian surface.

CLAIM 3.4. *Let X be an Abelian variety and $f : X \rightarrow S$ be a fibre space. Then S is an Abelian variety.*

PROOF. Let F be a general fibre of f . Since K_F is trivial, F is an Abelian variety by [7, Theorem 12]. We consider the following exact sequence:

$$0 \rightarrow F \rightarrow X \rightarrow X/F \rightarrow 0$$

Then the quotient morphism is nothing but $X \rightarrow S$. Thus $X/F \cong S$ and S is an Abelian variety. \square

We go back to the proof of Corollary 1.5. Since S does not contain curves whose selfintersection number are negative and $\kappa(S) \leq 0$, S is a \mathbb{P}^1 -bundle over an elliptic curve or a hyperelliptic surface if $q(S) = 1$. Moreover, S is \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$ if $q(S) = 0$. From these classification, we obtain assertions of Colollary 1.5. \square

4. CLASSIFICATION OF SINGULAR FIBRES OVER CODIMENSION ONE POINTS

In this section, we classify singular fibres over codimension one points of a projective Lagrangian fibration.

THEOREM 4.1. *Let $f : X \rightarrow S$ be a projective Lagrangian fibration and D the discriminat locus of f . Assume that $\dim X = 2n$, S and D is smooth at s . Then there exists a polydisk $s \in \Delta^n$ of S and the restriction morphism $f_{\Delta^n} : X_{\Delta^n} \rightarrow \Delta^n$ has one of the following two properties:*

- (1) *There is a toroidal model \tilde{X} of type I or II (See Definition 2.9), an Abelian fibration $\bar{X} \rightarrow \bar{\Delta}^n$ and an action of a cyclic group G_2 (resp. G_1) on \tilde{X} (resp. \bar{X}) which satisfy the following diagram.*

$$\begin{array}{ccccccc} X_{\Delta^n} & \xrightarrow{\eta} & \bar{X}/G_1 & \xleftarrow{\nu_1} & \bar{X} & = & \tilde{X}/G_2 & \xleftarrow{\nu_2} & \tilde{X} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Delta^n & = & \bar{\Delta}^n/G_1 & \leftarrow & \bar{\Delta}^n & = & \tilde{\Delta}^n/G_2 & \leftarrow & \tilde{\Delta}^n, \end{array}$$

where η is isomorphism on the smooth locus of \bar{X}/G_1 and \bar{X}/G_1 is normal. Moreover ν_1 is étale in codimension one and ν_2 is étale.

- (2) *There is an Abelian fibration $\tilde{f} : \tilde{X} \rightarrow \tilde{\Delta}^n$ and an action of a cyclic group G on \tilde{X} which satisfy the following properties:*
- (a) *X and \tilde{X} satisfy the following diagram:*

$$\begin{array}{ccc} X_{\Delta^n} & = & \tilde{X}/G \xleftarrow{\tau} \tilde{X} \\ \downarrow & & \downarrow \qquad \downarrow \\ \Delta^n & = & \tilde{\Delta}^n/G \leftarrow \tilde{\Delta}^n, \end{array}$$

where τ is an étale morphism.

- (b) *For the discriminant locus \tilde{D} of \tilde{f} , $\tilde{f}^*\tilde{D}$ is a reduced divisor.*
- (c) *There is a birational morphism $\nu : W \rightarrow \tilde{X}$ such that $\tilde{f} \circ \nu : W \rightarrow \tilde{\Delta}^n$ satisfies the assumptions (1) and (2) of Lemma 2.10.*

REMARK 5. In the case that $\dim X = 2$, f is a minimal elliptic fibration and f satisfies the properties of Theorem 4.1(2) if the singular fibre of f is a Kodaira singular fibre of type II, III or IV. If the singular fibre of f is the other types, f satisfies the properties of Theorem 4.1(1).

Before the proof of Theorem 4.1, we investigate the monodormy around s .

LEMMA 4.2. *Let $f : X \rightarrow S$ be a projective Lagrangian fibration. Assume that the discriminant locus of f and S are smooth at s . Then there is a polydisk $s \in \Delta^n$ of S and the restriction $f_{\Delta^n} : X_{\Delta^n} := X \times_S \Delta^n \rightarrow \Delta^n$ has the following properties:*

- (1) *There is a toroidal model $f^+ : X^+ \rightarrow \tilde{\Delta}^n$ of type I or II and the action of a cyclic group G on X^+ which commutes with f^+ and satisfies the following diagram:*

$$\begin{array}{ccc} X_{\Delta^n} & \xrightarrow{\alpha} & X^+/G \\ \downarrow & & \downarrow \\ \Delta^n & = & \tilde{\Delta}^n/G, \end{array}$$

where α is a bimeromorphic map.

- (2) *Let \tilde{D} be the ramification locus of $\tilde{\Delta}^n \rightarrow \Delta^n$, Then*

$$\dim F^1 G_1^W H^1(X_{\tilde{s}}^+, \mathbb{C})^G \geq n - 1,$$

where $X_{\tilde{s}}^+$ be a fibre of f^+ over a point \tilde{s} of \tilde{D} .

PROOF. (1) By the assumption that the discriminant locus is smooth at s , we may assume that S and D is smooth. Let ω be a symplectic form on X and $\nu : Y \rightarrow X$ a birational morphism such that $E = (f \circ \nu)^*D$ is a simple normal crossing divisor. We denote each component of E by E_i .

CLAIM 4.3. *Let $E := \sum E_i$ be a simple normal crossing variety. Then*

$$\mathrm{Gr}_k^W H^k(E, \mathbb{C}) = \{(\alpha_i) \in \bigoplus H^k(E_i, \mathbb{C}); \alpha_i|_{E_i \cap E_j} = \alpha_j|_{E_i \cap E_j}\}.$$

PROOF. Let

$$E^{[k]} := \coprod_{i_0 < \dots < i_k} (E_{i_0} \cap \dots \cap E_{i_k})^\sim,$$

where \sim means the normalization. For an index set $I = \{i_0, \dots, i_k\}$, we define an inclusion δ_j^I

$$\delta_j^I : E_{i_0} \cap \dots \cap E_{i_k} \rightarrow E_{i_0} \cap \dots \cap E_{i_{j-1}} \cap E_{i_{j+1}} \cap \dots \cap E_{i_k} \quad (0 \leq j \leq k).$$

We consider the following spectral sequence [4, Chapter 4]:

$$E_1^{p,q} = H^q(E^{[p]}, \mathbb{C}) \implies E^{p+q} = H^{p+q}(E, \mathbb{C}),$$

where $D : E_1^{p,q} \rightarrow E_1^{p+1,q}$ is defined by the

$$\bigoplus_{|I|=p} \sum_{j=0}^p (-1)^j (\delta_j^I)^*.$$

Since this spectral sequence degenerates at E_2 level ([4, Chapter 4.8]), we deduce

$$\mathrm{Gr}_k^W(H^k(E, \mathbb{C})) = \mathrm{Ker}(\bigoplus_i H^k(E_i, \mathbb{C}) \xrightarrow{D} \bigoplus_{i < j} H^k((E_i \cap E_j)^\sim, \mathbb{C})).$$

Thus we obtain the assertion of Claim 4.3 from the definition of D . \square

CLAIM 4.4. *Let U be an open set of D such that $f \circ \nu|_{(f \circ \nu)^{-1}(U)}$ is a locally trivial deformation and every fibre of $f \circ \nu|_{(f \circ \nu)^{-1}(U)}$ is a simple normal crossing variety. Then every point s' of U ,*

$$\dim F^1 \mathrm{Gr}_1^W H^1(Y_{s'}, \mathbb{C}) \geq n - 1,$$

where $Y_{s'}$ is the fibre of $f \circ \nu$ at s' .

PROOF OF CLAIM. First we show that there is a morphism

$$(5) \quad (f \circ \nu)^* T_U \rightarrow \Omega_{E_i/U}^1$$

for the component $E_i \cap (f \circ \nu)^{-1}(U) \neq \emptyset$. By the choice of U , $E_i \rightarrow U$ is a smooth morphism. Thus there is a following exact sequence:

$$0 \rightarrow T_{E_i/U} \rightarrow T_{E_i} \rightarrow (f \circ \nu)^* T_U \rightarrow 0.$$

From the following diagram,

$$\begin{array}{ccccccc} 0 & \rightarrow & T_{E_i} & \rightarrow & T_Y & \rightarrow & N_{E_i/Y} \rightarrow 0 \\ & & & & \downarrow & \nu^*\omega & \\ 0 & \rightarrow & I_{E_i}/I_{E_i}^2 & \rightarrow & \Omega_Y^1 & \rightarrow & \Omega_{E_i}^1 \rightarrow 0, \end{array}$$

we define $T_{E_i} \rightarrow \Omega_{E_i}^1$ by $\nu^*\omega$. By [14, Theorem 1], the restriction $\nu^*\omega$ to each component of a fibre of $f \circ \nu$ is identically zero. Hence $\nu^*\omega(\alpha, *) = 0$ in $\Omega_{E_i/U}^1$ for every element $\alpha \in T_{E_i/U}$ and there is the morphism (5). We take a point $s' \in U$ and denote local parameters of U at s' by t_j , ($2 \leq j \leq n$). Let $E_{i,s'}$ be the fibre of $E_i \rightarrow U$ over s' . From the morphism (5), $\alpha_{ij} := \nu^*\omega(\partial/\partial t_j, *)|_{E_{i,s'}}$ defines an element of $H^0(E_{i,s'}, \Omega_{E_{i,s'}}^1)$. We will show that α_{ij} , ($2 \leq j \leq n$) are linearly independent in $H^0(E_{i,s'}, \Omega_{E_{i,s'}}^1)$ for non ν -exceptional E_i . Assume the contrary. There exists a linear combination $\gamma := \sum \lambda_j \partial/\partial t_j$ such that $\gamma \neq 0$ and $\nu^*\omega(\gamma, *)|_{E_{i,s'}} = 0$ in $H^0(E_{i,s'}, \Omega_{E_{i,s'}}^1)$. From the definition of U , $E_{i,s'}$ is not contain ν -exceptional locus if E_i is not ν -exceptional. Thus there is an open set V of $E_{i,s}$ such that $\nu^*\omega$ is nondegenerate over V . For every point $y \in V$,

$$T_{E_{i,s',y}} = \{\alpha \in T_{Y,y}; \nu^*\omega(\alpha, \beta) = 0, \quad \forall \beta \in T_{E_{i,s',y}}\},$$

because $\nu(E_{i,s'})$ is a Lagrangian subvariety. However, $\gamma \notin T_{E_{i,s',y}}$ and $\nu^*\omega(\gamma, *)|_{E_{i,s'}} = 0$ in $H^0(E_{i,s'}, \Omega_{E_{i,s'}}^1)$. That derives a contradiction. If $E_{i,s'} \cap E_{i',s'} \neq \emptyset$, $\alpha_{i,j} = \alpha_{i',j}$ on $E_{i,s'} \cap E_{i',s'}$ by definition. Thus the assertion of Claim 4.4 follows by Claim 4.3. \square

We go back to the proof of Lemma 4.2 (1). We choose a neighborhood $\Delta^n(t_1, \dots, t_n)$ of s such that D is defined by $t_1 = 0$ and $\Delta^n \cap U \neq \emptyset$. Let $(f \circ \nu)^*D = \sum e_i E_i$ and $e := \text{L.C.M.}\{e_i; E_i \cap (f \circ \nu)^{-1}(U) \neq \emptyset\}$. We consider a e -fold cyclic cover

$$d: \tilde{\Delta}^n \ni (u, t_2, \dots, t_n) \mapsto (t_1 = u^e, t_2, \dots, t_n) \in \Delta^n$$

and the normalization \hat{Y} of $Y \times_{\Delta^n} \tilde{\Delta}^n$. Since \hat{Y} is projective and $g^{-1}(u)$ has a reduced component for a general point u of $d^{-1}(D)$, the Abelian fibration $g: \hat{Y} \rightarrow \Delta^n$ satisfies the assumptions of Lemma 2.8 except the monodromy conditions. Let \tilde{T} be the monodromy matrix of $R^1 g_* \mathbb{C}$, where g° is smooth part of g . We will show that \tilde{T} is unipotent and the rank of $\tilde{T} - I$ is at most one. Let s' be a point of $\Delta^n \cap U$. We choose an unit disk Δ^1 in Δ^n such that $\Delta^1 \cap D = \{s'\}$ and put $\tilde{\Delta}^1 = d^{-1}(\Delta^1)$. We concentrate the restriction morphism

$$g_{\tilde{\Delta}^1}: \hat{Y}_{\tilde{\Delta}^1} := \hat{Y} \times_{\tilde{\Delta}^n} \tilde{\Delta}^1 \rightarrow \tilde{\Delta}^1,$$

because the monodromy matrix of $R^1(g_{\tilde{\Delta}^1})_* \mathbb{C}$ is \tilde{T} , where $g_{\tilde{\Delta}^1}^\circ$ is smooth part of $g_{\tilde{\Delta}^1}$. By [9, Theorem 11*], if we take a suitable resolution $\tilde{Y}_{\tilde{\Delta}^1}$ of $\hat{Y}_{\tilde{\Delta}^1}$, every fibre of $\tilde{g}_{\tilde{\Delta}^1}: \tilde{Y}_{\tilde{\Delta}^1} \rightarrow \tilde{\Delta}^1$ is reduced and simple normal crossing.

CLAIM 4.5. *Let $Z_{\tilde{\Delta}^1}$ be a resolution of $\hat{Y}_{\tilde{\Delta}^1}$. We denote the fibre of $f \circ \nu$ at s' by $Y_{s'}$ and the fibre of $Z_{\tilde{\Delta}^1} \rightarrow \tilde{\Delta}^1$ at $\tilde{s}' := d^{-1}(s')$ by $Z_{\tilde{\Delta}^1, \tilde{s}'}$. Then the induced morphism $H^1(Y_{s'}, \mathbb{C}) \rightarrow H^1(Z_{\tilde{\Delta}^1, \tilde{s}'}, \mathbb{C})$ is injection.*

PROOF OF CLAIM. Let $Y_{\Delta^1} := Y \times_{\Delta^n} \Delta^1$. Since Y_{Δ^1} and $Z_{\tilde{\Delta}^1}$ are deformation retract to $Y_{s'}$ and $Z_{\tilde{\Delta}^1, \tilde{s}'}$ respectively, it is enough to prove that $H^1(Y_{\Delta^1}, \mathbb{C}) \rightarrow H^1(Z_{\tilde{\Delta}^1}, \mathbb{C})$ is injective. From the isomorphism $\hat{Y}_{\tilde{\Delta}^1}/G \cong Y_{\Delta^1}$, $H^1(Y_{\Delta^1}, \mathbb{C}) \cong H^1(\hat{Y}_{\tilde{\Delta}^1}, \mathbb{C})^G$ and $H^1(Y_{\Delta^1}, \mathbb{C}) \rightarrow H^1(\hat{Y}_{\tilde{\Delta}^1}, \mathbb{C})$ is injection. Moreover $\hat{Y}_{\tilde{\Delta}^1}$ is a homology manifold because $\hat{Y}_{\tilde{\Delta}^1}$ has only quotient singularities. ([18, Proposition 1.4]). Hence $H^1(\hat{Y}_{\tilde{\Delta}^1}, \mathbb{C}) \rightarrow H^1(Z_{\tilde{\Delta}^1}, \mathbb{C})$ is injection by [3, Théorème 8.2.4]. \square

We go back to the proof of Lemma 4.2. From Claim 4.5 and Claim 4.4,

$$\dim F^1 \mathrm{Gr}_1^W H^1(\tilde{Y}_{\tilde{s}'}, \mathbb{C}) \geq n - 1.$$

Since \tilde{g} is a semistable degeneration, \tilde{T} is unipotent. We consider the Clemens-Schmid exact sequence.

$$0 \rightarrow \mathrm{Gr}_1^W(H^1(\tilde{Y}_{\tilde{\Delta}^1, \tilde{s}'}, \mathbb{C})) \rightarrow \mathrm{Gr}_1^W(\tilde{Y}_{\tilde{\Delta}^1, \tilde{u}}, \mathbb{C}) \xrightarrow{\tilde{T}-I} \mathrm{Gr}_1^W(\tilde{Y}_{\tilde{\Delta}^1, \tilde{u}}, \mathbb{C}),$$

where $\tilde{Y}_{\tilde{\Delta}^1, \tilde{u}}$ is a general fibre of $\tilde{g}_{\tilde{\Delta}^1}$. From the above exact sequence,

$$\dim \mathrm{Gr}_1^W H^1(\tilde{Y}_{\tilde{\Delta}^1, \tilde{u}}, \mathbb{C}) \geq 2n - 2,$$

Moreover, the limit Hodge structure of $H^1(\tilde{Y}_{\tilde{\Delta}^1, \tilde{u}}, \mathbb{C})$ is as follows:

$$0 \subset W_0 = \mathrm{Im}(\tilde{T} - I) \subset W_1 = \mathrm{Ker}(\tilde{T} - I) \subset W_2 = H^1(\tilde{Y}_{\tilde{\Delta}^1, \tilde{u}}).$$

Since $\dim W_0 = \dim W_2/W_1$, the rank of $\tilde{T} - I$ is at most one. Therefore \hat{Y} satisfies the assumptions of Proposition 2.1 and there exists a toroidal model $f^+ : X^+ \rightarrow \tilde{\Delta}^n$ of type I or II such that X^+ is bimeromorphic to \hat{Y} . The Galois group G of the covering d acts on \hat{Y} and therefore act on X^+ bimeromorphically. This action commutes with f^+ . By Lemma 2.8, the action of G is biholomorphic and X^+/G is bimeromorphic to X_{Δ^n} over Δ^n .

(2) Let T be the monodromy matrix of $R^1(f \circ \nu)_* \mathbb{C}$ around D and $\tilde{D} := d^{-1}(D)$. For the fibre of $X_{\tilde{s}^+}$ of f^+ over $\tilde{s} \in \tilde{D}$, the action G on $H^1(X_{\tilde{s}^+}, \mathbb{C})$ is determined by the monodromy matrix T . Thus the function

$$\mu(\tilde{s}) : \tilde{s} \mapsto \dim F^1 \mathrm{Gr}_1^W H^1(X_{\tilde{s}^+}, \mathbb{C})^G$$

is constant on \tilde{D} . For the proof of the assertion of Lemma 4.2 (2), it is enough to show $\mu(\tilde{s}') \geq n - 1$ for $\tilde{s}' = d^{-1}(s')$. We consider the restriction morphism $X_{\tilde{\Delta}^1}^+ := X^+ \times_{\tilde{\Delta}^n} \tilde{\Delta}^1 \rightarrow \tilde{\Delta}^1$. Then there is a G -equivariant bimeromorphic map $\hat{Y}_{\tilde{\Delta}^1} \dashrightarrow X_{\tilde{\Delta}^1}^+$. We take a G -equivariant resolution of indeterminacy $Z_{\tilde{\Delta}^1}$ of $\hat{Y}_{\tilde{\Delta}^1} \dashrightarrow X_{\tilde{\Delta}^1}^+$. By Claim 4.5, $H^1(Y_{\Delta^1, s'}, \mathbb{C}) \rightarrow H^1(Z_{\tilde{\Delta}^1, \tilde{s}'}, \mathbb{C})$ is injection. By Claim 4.4,

$\dim F^1 \mathrm{Gr}_1^W H^1(Z_{\tilde{\Delta}^1, \tilde{s}}, \mathbb{C})^G \geq n - 1$. Because $Z_{\tilde{\Delta}^1} \rightarrow X_{\tilde{\Delta}^1}^+$ is G -equivariant bimeromorphic morphism, there is a G -equivariant isomorphism $H^1(Z_{\tilde{\Delta}^1}, \mathbb{C}) \cong H^1(X_{\tilde{\Delta}^1}^+, \mathbb{C})$. Since $Z_{\tilde{\Delta}^1}$ and $X_{\tilde{\Delta}^1}^+$ are deformation retract to $Z_{\tilde{\Delta}^1, \tilde{s}}$ and $X_{\tilde{\Delta}^1, \tilde{s}}^+$, respectively,

$$F^1 \mathrm{Gr}_1^W H^1(Z_{\tilde{\Delta}^1, \tilde{s}}, \mathbb{C})^G \cong F^1 \mathrm{Gr}_1^W H^1(X_{\tilde{s}}^+, \mathbb{C})^G.$$

Therefore we obtain $\mu(\tilde{s}') \geq n - 1$. \square

PROOF OF THEOREM 4.1. Let $f : X \rightarrow S$ be a projective Lagrangian fibration. By Lemma 4.2, there exists a toroidal model $f^+ : X^+ \rightarrow \tilde{\Delta}^n$ and an action of a cyclic group G . The assertions of Theorem 4.1 follows the following Proposition.

PROPOSITION 4.6. (1) *If f^+ is a toroidal model of type I, then $f^{-1}(s)$ satisfies the properties of Theorem 4.1 (1) or (2).*

(2) *If f^+ is a toroidal model of type II, then $f^{-1}(s)$ is satisfies the properties of Theorem 4.1 (1).*

PROOF OF PROPOSITION 4.6. (1) We begin with to prove that the representation $G \rightarrow H^0(X_{\tilde{s}}^+, \mathbb{C})$ is not trivial, where $X_{\tilde{s}}^+$ is the fibre of f^+ at \tilde{s} . Assume the contrary. Since every fibre of f^+ is an Abelian variety, the action of G on X^+ is fixed point free. Thus X^+/G is smooth and $K_{X^+/G}$ is numerically trivial. Since X^+/G has no rational curve, X^+/G is the unique relative minimal model. On the contrary, X_{Δ^n} is a relative minimal model. Therefore $X_{\Delta^n} \cong X^+/G$. However, $K_{X^+/G} \not\sim 0$ because K_{X^+} is not G -invariant. This derives a contradiction, because $K_{X_{\Delta^n}} \sim 0$. We need the following Lemma to prove Proposition 4.6 (1).

LEMMA 4.7. *Let $X_D^+ := (f^+)^{-1}(\tilde{D})$, where \tilde{D} is the ramification locus of $\tilde{\Delta}^n \rightarrow \Delta^n$. Then $X_D^+ \rightarrow \tilde{D}$ decompose a G -equivariant smooth elliptic fibration $X_D^+ \rightarrow A'$ and a G -equivariant smooth Abelian fibration $A' \rightarrow \tilde{D}$. Moreover they satisfy the following diagram:*

$$\begin{array}{ccccc} X_D^+ & \rightarrow & A' & \rightarrow & \tilde{D} \\ \downarrow & & \downarrow & & \downarrow \\ X_D^+/G & \rightarrow & A & \rightarrow & D, \end{array}$$

where D is the branch locus of $\tilde{\Delta}^n \rightarrow \Delta^n$, $A \rightarrow D$ are smooth $n - 1$ dimensional Abelian fibration and $A'/G \cong A$.

PROOF. Let F be a fibre of $X_D^+ \rightarrow \tilde{D}$. First we show that F/G is smooth and $q(F/G) = n - 1$. Let T be the representation matrix of $\rho : G \rightarrow \mathrm{Aut} H^0(F, \Omega_F^1)$. Since G is a finite cyclic group and Lemma 4.2 (2), $T = \mathrm{diag}(\zeta, 1, \dots, 1)$ under suitable coordinate of $H^0(F, \Omega_F^1)$. Because F is an Abelian variety, the action of G is written $(z_1, z_2, \dots, z_n) \mapsto (\zeta z_1, z_2, \dots, z_n)$ around a fixed point p of F , where (z_1, \dots, z_n) is a local coordinate of p and ζ is a n -th root of unity. Note that $\zeta \neq 1$

because ρ is not trivial representation. The branch locus of the quotient map $F \rightarrow F/G$ are smooth divisors of F and F/G is smooth. Since $\dim H^0(F, \Omega_F^1)^G = n - 1$, $q(F/G) = n - 1$.

Next we consider relative Jacobian of $X_D^+/G \rightarrow D$. Since $q(F/G) = n - 1$, there is a $n - 1$ -dimensional smooth Abelian fibration $A \rightarrow D$ which factor $X_D^+/G \rightarrow D$. We prove that $\alpha : X_D^+/G \rightarrow A$ is surjective and every fibre is connected. We consider the following diagram

$$\begin{array}{ccc} X_D^+ & \rightarrow & B_1 \\ \downarrow & & \downarrow \\ X_D^+/G & \rightarrow & B_2 \rightarrow A, \end{array}$$

where $X_D^+/G \rightarrow B_2$ is the stein factorization of α and $X_D^+ \rightarrow B_1$ is the stein factorization of $X_D^+ \rightarrow B_2$. Let A_p (resp. $B_{1,p}, B_{2,p}$) be the fibre of $A \rightarrow D$ (resp. $B_1 \rightarrow D, B_2 \rightarrow D$) at p . Note that the fibre X_p^+ of $X_D^+ \rightarrow \tilde{D}$ at p is an Abelian variety and $X_p^+ \rightarrow B_{1,p}$ is surjective and connected fibre. By Claim 3.4, $B_{1,p}$ is an Abelian variety. Since $B_{1,p} \rightarrow B_{2,p}$ is a finite morphism, $\kappa(B_{2,p}) \leq \kappa(B_{1,p}) = 0$ by [7, Corollary 9]. From [7, Theorem 13], there is an étale morphism $\tilde{B}_{2,p} \rightarrow B_{2,p}$ and $\tilde{B}_{2,p}$ is isomorphic to the product of an Abelian variety and a variety of general type. Thus $0 \leq \kappa(\tilde{B}_{2,p}) = \kappa(B_{2,p})$. Therefore $\kappa(\tilde{B}_{2,p}) = \kappa(B_{2,p}) = 0$, $\tilde{B}_{2,p}$ is an Abelian variety and $K_{B_{2,p}}$ is numerically trivial. Since $B_{2,p} \rightarrow A_p$ is finite and A_p is an Abelian variety, $B_{2,p}$ is an Abelian variety. By the universal properties of relative Jacobian, $B_2 \cong A$. The stein factorization $X_D^+ \rightarrow A'$ of $X_D^+ \rightarrow A$ is the desired morphism. \square

We go back to the proof of Proposition 4.6 (1). By Lemma 4.7, there is a G -equivariant smooth Abelian fibration $\pi' : A' \rightarrow \tilde{D}$. Since $A'/G \cong A$, the action of G on A' is a translation on each fibre of π' . Let g be a generator of G and m the smallest integer such that the action of g^m on each fibre of π' is trivial.

First we consider the case that $H = \{1\}$. In this case, the action of G is fix point free. Hence X^+/G is smooth. Moreover X^+/G is the unique relative minimal model over Δ^n since it has no rational curves. On the contrary, X_{Δ^n} is a relative minimal model over Δ^n , $X_{\Delta^n} \cong X^+/G$. Thus X satisfies the properties of Theorem 4.1 (1) if we put $\tilde{X} = X^+$, $G_1 = \{1\}$ and $G_2 = G$.

Next we consider the case that $H \neq \{1\}$. Since the action of H on each fibre of π' is trivial and $X_D^+ \rightarrow A'$ is a smooth elliptic fibration, every singular locus of X^+/H forms a multisection of $X_D^+/H \rightarrow A'$ and $X_D^+/H \rightarrow A'$ is a \mathbb{P}^1 -bundle. For every singular point p of X^+/H , $(p, X^+/H) \cong (\text{surface cyclic quotient singularity}) \times (\mathbb{C}^{2n-2}, 0)$. The list of surface cyclic quotient singularities which occur above is found in [1, Table 5 (pp 157)]. By definition, the action of G/H on each fibre of π' is non trivial translation. Thus the action of G/H on X^+/H is fix point free and the

quotient morphism $X^+/H \rightarrow X^+/G$ is an étale morphism. Every singular locus also forms a multisection of $X_D^+/G \rightarrow A$ and $X_D^+/G \rightarrow A$ is a \mathbb{P}^1 -bundle. For every singular point p of X^+/G , $(p, X^+/G) \cong (\text{surface cyclic quotient singularity}) \times (\mathbb{C}^{2n-2}, 0)$. We will construct a suitable resolution X^+/G and a relative minimal model over Δ^n according to singularities of X^+/G .

If every singular point p of X^+/G , $(p, X^+/G) \cong (\text{Du Val singularity}) \times (\mathbb{C}^{2n-2}, 0)$, we take the minimal resolution $\eta : X' \rightarrow X^+/G$ by using the minimal resolution of Du Val singularities. Then X' is a relative minimal model over Δ^n . The morphism $\eta^*(X_D^+/G) \rightarrow A$ is a locally trivial deformation whose fibre is a Kodaira singular fibre of type I_0^* , II^* , III^* or IV^* . Combining that $A \rightarrow D$ is a smooth Abelian fibration, we conclude that every irreducible component of $\eta^*(X_D^+/G)$ is a \mathbb{P}^1 -bundle over a smooth Abelian fibration. By similar argument in Lemma 2.8, X' is the unique minimal model over Δ^n . Hence $X_{\Delta^n} \cong X'$. It is verified that X satisfies the properties of Theorem 4.1 (1) by $\tilde{X} = \bar{X} = X^+$, $G_1 = G$ and $G_2 = \{1\}$.

If some singular point p of X^+/G , $(p, X^+/G) \not\cong (\text{Du Val singularity}) \times (\mathbb{C}^{2n-2}, 0)$, we consider the minimal resolution $w : W \rightarrow X^+/H$, which is obtained from the minimal resolution of surface quotient singularities. Then $w^*(X_D^+/H) \rightarrow A'$ is a locally trivial deformation whose fibre is a tree of \mathbb{P}^1 . (cf. [1, pp 158]) Since $A' \rightarrow \tilde{D}/H$ is a smooth Abelian fibration, $w^*(X_D^+/H)$ is a simple normal crossing divisor and the morphism $w^*(X_D^+/H) \rightarrow \tilde{D}/H$ is a locally trivial deformation. Therefore W satisfies the assumption (1) and (2) of Lemma 2.10. Every irreducible component of $w^*(X_D^+/H)$ is a \mathbb{P}^1 -bundle over a smooth Abelian fibration. By similar argument in Lemma 2.8, W is the unique minimal resolution of X^+/H . Thus the action of G/H on X^+/G can be lifted on W and $W \rightarrow X^+/H$ is G/H -equivariant morphism. Hence $w^*(X_D^+/H) \rightarrow A'$ is G/H -equivariant. By contracting \mathbb{P}^1 -bundles along its ruling, (cf. [1, pp 158]) we obtain a G/H -equivariant relative minimal model \tilde{X} of W over $\tilde{\Delta}^n/H$. Let $\tilde{f} : \tilde{X} \rightarrow \tilde{\Delta}^n/H$ and $\tilde{X}_{\tilde{D}/H} := \tilde{f}^{-1}(\tilde{D}/H)$. By construction, \tilde{X} is smooth and every singular fibre of \tilde{X} is reduced. The morphism $\tilde{X}_{\tilde{D}/H} \rightarrow A'$ is a G/H -equivariant locally trivial deformation whose fibre is a Kodaira singular fibre of type II , III or IV . Thus the normalization of every irreducible component of $\tilde{X}_{\tilde{D}/H}$ is a \mathbb{P}^1 -bundle over a smooth Abelian fibration. By similar argument in Lemma 2.8, \tilde{X} is the unique minimal model. Since the action of G/H on A' is non trivial translation, the action of G/H on \tilde{X} is fixed point free. Thus $\tilde{X} \rightarrow \tilde{X}/(G/H)$ is an étale morphism and $\tilde{X}/(G/H)$ is a relative minimal model. Moreover, $\tilde{X}/(G/H)$ is the unique relative minimal model because \tilde{X} is so. Hence $\tilde{X}/(G/H) \cong X_{\Delta^n}$. Therefore \tilde{X} , X and W satisfies the properties of Theorem 4.1 (2). We finish the proof of Proposition 4.6 (1).

(2) First we prove the following Lemma.

LEMMA 4.8. *Let \tilde{D} be the discriminat locus of \tilde{f} and $\tilde{f}^*\tilde{D} = \sum E_i$. We define the subgroup H of G as follows:*

$$H := \{g \in G; g(E_i) = E_i \text{ for all } i.\}$$

Then there is the unique minimal relative minimal model $\bar{f} : \bar{X} \rightarrow \tilde{\Delta}^n/H$ of $X^+/H \rightarrow \tilde{\Delta}^n/H$ which have the properties of Proposition 2.1 (1), (2) and (3). Moreover

- (1) *There is an action of G/H on \bar{X} . If an element g of G/H preserves every irreducible component of every singular fibre, then $g = 1$.*
- (2) *For every point s of the discriminat locus of \bar{f} ,*

$$\dim F^1 \text{Gr}^1 H^1(\bar{X}_s, \mathbb{C}) \geq n - 1,$$

where \bar{X}_s is the fibre of \bar{f} at s .

- (3) *There exists an étale morphism $\nu : \tilde{X} \rightarrow \bar{X}$ and an cyclic group G' such that \tilde{X} is a toroidal model of type II and $\tilde{X}/G' \cong \bar{X}$.*

PROOF. We begin with investigation of the action of H on each E_i . Let $X_{\tilde{s}}^+$ be the fibre of f^+ over $\tilde{s} \in \tilde{D}$. We denote each irreducible component of $X_{\tilde{s}}^+$ by $E_{i,\tilde{s}}$ and $F_{ij,\tilde{s}} := E_{i,\tilde{s}} \cap E_{j,\tilde{s}}$. From the following Claim, the action of H on $H^0(E_{i,\tilde{s}}, \Omega_{E_{i,\tilde{s}}}^1)$ and $H^0(F_{ij,\tilde{s}}, \Omega_{F_{ij,\tilde{s}}}^1)$ are trivial.

CLAIM 4.9. *Let $E = \sum E_{i,\tilde{s}}$ be a cycle of several copies of a \mathbb{P}^1 -bundle over an Abelian variety such that $F_{ij,\tilde{s}} := E_{i,\tilde{s}} \cap E_{j,\tilde{s}}$ forms a section of the ruling of $E_{i,\tilde{s}}$ and $E_{j,\tilde{s}}$. Assume that there is an action of a cyclic group G on E such that $\dim F^1 \text{Gr}_1^W H^1(E, \mathbb{C})^G \geq n - 1$. Then*

- (1) *If $F_{ij,\tilde{s}}$ is G -stable, then every element of $H^0(F_{ij,\tilde{s}}, \Omega_{F_{ij,\tilde{s}}}^1)$ is G -invariant.*
- (2) *If $E_{i,\tilde{s}}$ is G -stable, then every element of $H^0(E_{i,\tilde{s}}, \Omega_{E_{i,\tilde{s}}}^1)$ is G -invariant.*

PROOF. Assume that some $F_{ij,\tilde{s}}$ is G -stable and there exists a non G -invariant element of $H^0(F_{ij,\tilde{s}}, \Omega_{F_{ij,\tilde{s}}}^1)$. Since every $E_{i,\tilde{s}}$ is a \mathbb{P}^1 -bundle over an Abelian variety and every $F_{ij,\tilde{s}}$ forms a section of the ruling, there is a G -equivariant isomorphism $F^1 \text{Gr}_1^W H^1(E, \mathbb{C}) \cong H^0(F_{ij,\tilde{s}}, \Omega_{F_{ij,\tilde{s}}}^1)$ by Claim 4.3. Thus there exists a non G -invariant element in $F^1 \text{Gr}_1^W H^1(E, \mathbb{C})$. On the contrary, $\dim F^1 \text{Gr}_1^W H^1(E, \mathbb{C}) = n - 1$. Hence $F^1 \text{Gr}_1^W H^1(E, \mathbb{C})^G = F^1 \text{Gr}_1^W H^1(E, \mathbb{C})$. That is a contradiction. If some $E_{i,\tilde{s}}$ is G -stable, we obtain the assertion of Claim by similar argument. \square

We go back to the proof of Lemma 4.8. Since $F_{ij,\tilde{s}}$ is an Abelian variety, the action of H is translation. If the action of H on $F_{ij,\tilde{s}}$ is not trivial, the action of H is fixed point free. We put $\bar{X} := X^+/H$. It is easy to check that X^+/H satisfies the assertions of Lemma.

We consider the case that the action of H on $F_{ij,\tilde{s}}$ is trivial. In this case, every point of $F_{ij} := E_i \cap E_j$ is fixed by H . We choose a point $p \in F_{ij}$ and a local

coordinate z_i , ($1 \leq i \leq 2n$) at p such that E_i (resp. E_j) is defined by $z_1 = 0$. (resp. $z_2 = 0$.) Since E_i and E_j are stable under the action of H , the action of H can be written as

$$(z_1, z_2, z_3, \dots, z_{2n}) \mapsto (az_1, bz_2, z_3, \dots, z_{2n}),$$

where $a, b \in \mathbb{C}^*$. Thus for every singular point q of X^+/H ,

$$(q, X^+/H) \cong (\text{surface cyclic quotient singularity}) \times (\mathbb{C}^{2n-2}, 0).$$

The morphism $((f^+)^*\tilde{D})/H \rightarrow \tilde{D}/H$ is a locally trivial deformation. Every irreducible component of every fibre is a \mathbb{P}^1 -bundle over an Abelian variety. The singular locus of X^+/H is contained in $\cup_{i < j} (F_{ij}/H)$. We take the minimal resolution $\nu : Z_0 \rightarrow X^+/H$ by using the minimal resolution of a surface cyclic quotient singularity. Then the morphism $\nu^*((f^+)^*\tilde{D})/H \rightarrow \tilde{D}/H$ is a locally trivial deformation. Since exceptional locus of the minimal resolution of a surface cyclic quotient singularity is a chain of \mathbb{P}^1 , every fibre of $\nu^*((f^+)^*\tilde{D})/H \rightarrow \tilde{D}/H$ is a cycle of several copies of a \mathbb{P}^1 -bundle over an Abelian variety. Hence Z_0 satisfies the properties (1) and (2) of Proposition 2.1. Every irreducible component of $\nu^*((f^+)^*\tilde{D})/H$ is a \mathbb{P}^1 -bundle over a smooth Abelian fibration because $\pi(\tilde{D}/H) = \{1\}$. By similar argument in Lemma 2.8, Z_0 is the unique minimal resolution of X^+/H . Therefore the action of G/H can be lifted on Z_0 . Let $z_0 : Z_0 \rightarrow Z_1$ be a G/H -equivariant extremal contraction. Let ℓ be a curve which is contracted by z_0 . We denote the irreducible component of $\nu^*((f^+)^*\tilde{D}/H)$ which contains ℓ by E_0 . Since E_0 is a \mathbb{P}^1 -bundle over a smooth Abelian fibration, ℓ is a fibre of the ruling of E_0 . Thus z_0 is the contraction of \mathbb{P}^1 -bundles along its ruling. Hence Z_1 is smooth and it satisfies the properties (1) and (2) of Proposition 2.1. Iterating this process, we obtain a G/H -equivariant minimal model \bar{X} . By construction, \bar{X} has the properties (1), (2) and (3) of Proposition 2.1. By Lemma 2.8, \bar{X} is the unique relative minimal model. Let \bar{g} be an element of G/H . By the definition of H , $\bar{g} = 1$ if every irreducible component of singular fibres of $Z_0 \rightarrow \tilde{\Delta}^n/H$ are stable under the action of \bar{g} . Since at least one \bar{g} -orbit is not contracted in the contracting process $Z_0 \rightarrow \bar{X}$, \bar{X} satisfies the assertion (1) of Lemma.

For the proof of (2) and (3) of Lemma, we consider the restriction of \bar{X} . Let C be a smooth curve of $\tilde{\Delta}^n/H$ which is stable under the action of G/H and which intersects transversally with the discriminant locus of \bar{f} at s . We consider the restrictions $(X^+/H)_C := X^+/H \times_{\tilde{\Delta}^n/H} C$, $(Z_0)_C := Z_0 \times_{\tilde{\Delta}^n/H} C$ and $\bar{X}_C := \bar{X} \times_{\tilde{\Delta}^n/H} C$. Since X^+ satisfies the property (2) of Proposition 2.1, $(X^+/H)_C$ has only quotient singularities. Hence there is an injection $H^1((X^+/H)_C, \mathbb{C}) \rightarrow H^1((Z_0)_C, \mathbb{C})$. Thus $H^1((X^+/H)_s, \mathbb{C}) \rightarrow H^1((Z_0)_s, \mathbb{C})$ is an injection because $(X^+/H)_C$ and $(Z_0)_C$

are deformation retract to $(X^+/H)_s$ and $(Z_0)_s$ respectively. There is the G/H -equivariant bimeromorphic morphism $(Z_0)_C \rightarrow (X^+/H)_C$. Moreover

$$F^1 \mathrm{Gr}_1^W H^1((X^+/H)_s, \mathbb{C})^{G/H} \cong F^1 \mathrm{Gr}_1^W H^1(X_s^+, \mathbb{C})^G.$$

We obtain

$$\dim F^1 \mathrm{Gr}_1^W H^1((Z_0)_s, \mathbb{C})^{G/H} \geq n - 1,$$

from Lemma 4.2 (2). Since $(Z_0)_C \rightarrow \bar{X}_C$ is G/H -equivariant bimeromorphic morphism, there is a G/H -equivariant isomorphism $H^1(\bar{X}_C, \mathbb{C}) \rightarrow H^1((Z_0)_C, \mathbb{C})$. Hence $F^1 \mathrm{Gr}_1^W H^1((Z_0)_s, \mathbb{C})^{G/H} \cong F^1 \mathrm{Gr}_1^W H^1(\bar{X}_s, \mathbb{C})^{G/H}$ because \bar{X}_C is deformation retract to \bar{X}_s . We obtain the assertion (2) of Lemma.

For the proof of the assertion (3) of Lemma, we need the following Claim.

CLAIM 4.10. *Let $\bar{f} : \bar{X} \rightarrow \tilde{\Delta}^n/H$ and $\bar{f}^*(\tilde{D}/H) = \sum e_i \bar{E}_i$. Then $e_i = m$ for all i .*

PROOF. Assume the contrary. Let \bar{E}_0 be the component such that e_0 attains minimum value of e_i and \bar{E}_i , ($i = 1, 2$) next components. We may assume that $e_1 > e_0$. We consider the restriction \bar{X}_C and denote the restriction \bar{E}_i to \bar{X}_C by $\bar{E}_{C,i}$. Since \bar{f} satisfies the properties (1), (2) and (3) of Proposition 2.1, each $\bar{E}_{C,i}$ is a \mathbb{P}^1 -bundle over an Abelian variety and $K_{\bar{E}_{C,0}} \sim (K_{\bar{X}_C} + \bar{E}_{C,0})|_{\bar{E}_{C,0}} \sim \bar{E}_{C,0}|_{\bar{E}_{C,0}}$. Let ℓ be a fibre of the ruling of $\bar{E}_{C,0}$. Since each intersection $\bar{E}_{C,i} \cap \bar{E}_{C,j}$ forms a section of the ruling of $\bar{E}_{C,i}$.

$$\left(\sum e_i \bar{E}_{C,i} \right) \ell = e_1 + e_2 + e_0 \bar{E}_{C,0} \cdot \ell = 0.$$

Because ℓ is a fibre of the ruling of $\bar{E}_{C,0}$, $K_{\bar{E}_{C,0}} \cdot \ell = -2$. From the assumption $e_1 > e_0$ and $e_2 \geq e_0$, this derives a contradiction. \square

We choose a coordinate (t_1, \dots, t_n) of $\tilde{\Delta}^n/H$ such that \tilde{D}/H is defined by $t_1 = 0$. Let $d : \tilde{\Delta}^n(s_1, t_2, \dots, t_n) \rightarrow (\tilde{\Delta}^n/H)(t_1, \dots, t_n)$ be the m -fold cyclic covering defined by $s_1^m = t_1$, G' the galois group of d and $\tilde{X} := \bar{X} \times_{\tilde{\Delta}^n/H} \tilde{\Delta}^n$. Then $\tilde{X} \rightarrow \bar{X}$ is étale, \tilde{X} is smooth and every fibre is reduced. (cf. [18, Lemma 2.2]) We complete the proof of Lemma. \square

We go back to the proof of Proposition 4.6 (2). Let us consider the action of G/H on \bar{X} , which is constructed in Lemma 4.8. Since the dual graph Γ of every singular fibre of \bar{X} is the Dynkin diagram of type \tilde{A}_n , the action of G/H on Γ is rotation or reflection. If the action of G/H is rotation, the action of G/H on \bar{X} is fixed point free by the assertion (1) of Lemma 4.8. The quotient $\bar{X}/(G/H)$ is the unique minimal model because \bar{X} is so by Lemma 4.8. Hence $\bar{X}/(G/H) \cong X_{\Delta^n}$. Therefore X , \bar{X} and \tilde{X} satisfy the properties of Theorem 4.1 (1).

We consider the case that the action of G/H on Γ is reflection. If the action of G/H is fixed point free, we prove that $\bar{X}/(G/H) \cong X$ and X satisfies the properties (1) of Theorem 4.1 by similar argument in the case that the action of

G/H on Γ is rotation. Thus we assume that G/H has fixed points. We use same notations as in Lemma 4.8 and Claim 4.10. Since \bar{X} satisfies the properties (1) and (2) of Proposition 2.1, the morphism $\bar{f}^*(\tilde{D}/H) \rightarrow \tilde{D}/H$ is a locally trivial deformation and every fibre of $\bar{f}^*(\tilde{D}/H) \rightarrow \tilde{D}/H$ is a cycle of several copies of a \mathbb{P}^1 -bundle over an Abelian variety. Since $\pi_1(\tilde{D}/H) = \{1\}$, each \bar{E}_i is a \mathbb{P}^1 -bundle over a smooth Abelian fibration. From the assumption that the action G/H on Γ is reflection, there are two G/H -stable component in $\bar{f}^*(\tilde{D}/H)$. If \bar{E}_i is G/H -stable, every one form on $\bar{E}_{i,s}$ is G/H -invariant by Lemma 4.8 (2) and Claim 4.9, where $\bar{E}_{i,s}$ is a fibre of $\bar{E}_i \rightarrow \tilde{D}/H$. Since $\bar{E}_{i,s}$ is a \mathbb{P}^1 -bundle over an Abelian variety, every fibre of the ruling of \bar{E}_i is G/H -stable and the fixed locus on \bar{E}_i forms multisection of the ruling of \bar{E}_i . If $\bar{F}_{ij} := \bar{E}_i \cap \bar{E}_j$ is G/H -stable, every one form of $\bar{F}_{ij,s} := \bar{E}_{i,s} \cap \bar{E}_{j,s}$ is G/H -invariant by Lemma 4.8 (2) and Claim 4.9. We obtain that the action of G/H on \bar{F}_{ij} is trivial because $\bar{F}_{ij,s}$ is an Abelian variety and we assume that there exists a fixed point. Since \bar{F}_{ij} forms a section of the ruling of \bar{E}_i and \bar{E}_j , every connected component of the fixed locus of \bar{X} forms a multisection of the ruling of some \bar{E}_i . For every singular point p of $\bar{X}/(G/H)$, $(p, \bar{X}/(G/H)) \cong (A_1\text{-singularity}) \times (\mathbb{C}^{2n-2}, 0)$. We obtain a relative minimal model $\bar{f}^+ : \bar{X}^+ \rightarrow \Delta^n$ of $\bar{X}/(G/H)$ over $\tilde{\Delta}^n/G = \Delta^n$ by blowing up along singular locus. Let D be the discriminant locus of \bar{f}^+ . Since every irreducible component of $(\bar{f}^+)^*(D)$ is a \mathbb{P}^1 -bundle over a smooth Abelian fibration, we obtain that \bar{X}^+ is the unique relative minimal model by similar argument in Lemma 2.8. Hence $X_{\Delta^n} \cong \bar{X}^+$. From Lemma 4.8, there is an étale covering $\tilde{X} \rightarrow \bar{X}$ such that \tilde{X} is a toroidal model of type II. Thus X satisfies the properties of Theorem 4.1 (1). \square

We complete the proof of Theorem 4.1. \square

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