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The following set of notes has been taken from Differential Geometry II course at UIUC with Professor Christopher J. Leininger. Some modifications have been made and additional details have been provided but if you see any typos or if you have any questions, feel free to submit them to mim2 (at) math (dot) uiuc (dot) edu.

Let $g = <\,,\,>$ be an oriented Riemann metric on a manifold $M = M^n$ and $\Delta : \Omega^q(M) \to \Omega^q(M)$ be the Laplace operator. The Laplacian is a second order differential operator which is a generalization of $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^n$.

Let $\mathcal{H}^q(M) = \ker(\Delta) \cap \Omega^q(M)$ be harmonic q-forms.

Corollary of Hodge Theorem. For a compact manifold M, $\Omega^q(M) = \Delta(\Omega^q(M)) \bigoplus \mathcal{H}^q(M)$ and $\mathcal{H}^q(M) \cong H^q_{dR}(M)$.

Discussion of Δ.

The pairing $<\,,\,>_m$ induces a metric on the cotangent space $T^*_m M$ called $<\,,\,>_m$ so that the isomorphism

$$T_m M \to T^*_m M \tag{1}$$

$$v \mapsto < v, \cdot > = \tilde{v} \tag{2}$$

is an isometry.

Note: If $x^1, ..., x^n$ are coordinates on an open neighborhood $U \subseteq M$, then $<,>$ is described by (g_{ij}) with $g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$. We also obtain $g^{ij} = \left\langle dx^i, dx^j \right\rangle$ and $g^{ij} g_{jk} = \delta^i_k$.

The pairing $<\,,\,>_m$ also induces $<\,,\,>_m$ on $\wedge^q T^*_m M$ by

$$\left\langle \phi^1 \wedge ... \wedge \phi^q, \psi^1 \wedge ... \wedge \psi^q \right\rangle = \det\left(\left\langle \phi^i, \psi^j \right\rangle \right),$$

which we extend linearly. We view the pairing as an inner product on $\wedge^p T^*_m M$ with $\wedge^p T^*_m M \perp \wedge^q T^*_m M$ whenever $p \neq q$.

$*$-operator.

Let $M = M^n$ be an n-dimensional manifold. Let $* : \Omega^*(M) \to \Omega^*(M)$. More specifically, $* : \Omega^q(M) \to \Omega^{n-q}(M)$. Since we define the $*$-operator pointwise, $* : \wedge^q T_* M \to \wedge^{n-q} T_* M$ can be defined and $\wedge^q T_* M \cong \wedge^{n-q} T_* M$ are isomorphic as vector spaces.

$*$ for cases when $q = 0, n$. The pairing induces a volume form $\nu \in \Omega^n(M)$. Let $\nu_m = e^1 \wedge ... \wedge e^n \in \wedge^n(T^*_m M)$ with $e^1, ..., e^n$ being oriented orthonormal basis for $T^*_m M$. Then $* : \wedge^0 T^*_m M = \mathbb{R} \to \wedge^n T_m M$ is a map so that $*1 = \nu$ and $*\nu = 1$.

For $0 < q < n$, the $*$-map is defined as $*(e^1 \wedge ... \wedge e^q) = e^{q+1} \wedge ... \wedge e^n$. Alternatively, we consider the pairing $<\,,\,>_m$: $\wedge^q T^*_m M \times \wedge^q T^*_m M \to \mathbb{R}$ and $*\wedge : \wedge^q T^*_m M \times \wedge^{n-q} T^*_m M \to \mathbb{R}$ that maps

$$(\eta_m, \mu_m) \mapsto *(\eta_m \wedge \mu_m).$$

This gives us an isomorphism between $\wedge^q T_m^* M$ and $\wedge^{n-q} T_m^* M$ and this is considered to be the $*$-map.

Note: $\langle \mu_m, \eta_m \rangle = * (\mu_m \wedge * \eta_m) = * (\eta_m \wedge * \mu_m)$.

Example. Let $\omega \in \Omega^1(M)$ and $\tilde{\omega}$ is dual to ω. We let $*\omega = \iota_{\tilde{\omega}} \nu$, which is the contraction of the dual vector into volume form ν. So for $\omega_m = te^1$ for some t,

$$*\omega_m = *te^1 = t * e^1 = te^2 \wedge ... \wedge e^n.$$

On the other hand, we obtain

$$\iota_{\tilde{\omega}_m} \nu_m = \iota_{te_1} \nu_m = t\iota_{e_1} \nu_m \tag{3}$$

$$= t\iota_{e_1}(e^1 \wedge ... \wedge e^n) \tag{4}$$

$$= t(e^2 \wedge ... \wedge e^n) \tag{5}$$

so both sides make sense.

We see that ν_m is the unique representative of orientation with $||\nu_m|| = 1$ and $** = (-1)^{p(n-p)}$ because for example, given an orthonormal basis $e^1, ..., e^n$ on $T_m^* M$, applying the $*$-operator once gives $*(e^1 \wedge ... \wedge e^p) = e^{p+1} \wedge ... \wedge e^n$ while applying it twice gives us $**(e^1 \wedge ... \wedge e^p) = *(e^{p+1} \wedge ... \wedge e^n) = (-1)^{p(n-p)} e^1 \wedge ... \wedge e^p$.

Now let's define $\delta : \Omega^q(M) \to \Omega^{q-1}(M)$ where $\delta = (-1)^{n(q+1)+1} * d*$ and $\Delta = d\delta + \delta d$.

Try checking that $\Delta : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is $\Delta = -\sum_{i=1}^n \partial^2 / \partial x^{i2}$.

Now what is the Laplacian Δ_m on $C^\infty(M) = \Omega^0(M)$?

$$\Delta(f) = (\delta d + d\delta)(f) \tag{6}$$

$$= \delta df + d\delta f = \delta df \quad \text{because } d\delta = 0 \tag{7}$$

$$= (-1)^{n(2)+1} * d(*df) \tag{8}$$

$$= (-1)^{2n+1} * d\iota_{grad(f)}(\nu) \quad \text{since } grad(f) = \nabla f = \tilde{d}\tilde{f} \tag{9}$$

$$= - * d\iota_{grad(f)}(\nu) \tag{10}$$

$$= - * L_{grad(f)}(\nu) \quad \text{by Cartan } L_\xi \nu = div(\xi)\nu = (d\iota_\xi + \iota_\xi d)\,\nu \tag{11}$$

$$= - * div(grad(f))\nu \tag{12}$$

$$= -div(grad(f)) \tag{13}$$

Note that the Cartan form tells us how flow changes the volume form. So for harmonic q-forms $\mathcal{H}^q(M) = \ker(\Delta) \cap \Omega^q(M)$ and for $f \in \mathcal{H}^0(M)$, $\Delta f = 0$ if and only if the gradient flow preserves volume. Also on a compact manifold, $f \in \mathcal{H}^q(M) = H^q_{dR}(M)$ is a locally constant q-form.

For a 1-form ω, $-div(\tilde{w}) = \delta\omega$, and $\delta\omega = 0$ if and only if $\tilde{w}$ preserves volume. Now for a 1-form to be harmonic, $\Delta\omega = 0 \Leftrightarrow d\omega = 0$ (it is closed) and $\delta\omega = 0$ (it is co-closed). By Frobenius Theorem we say that there is foliation. That is, let M be an n-dimensional manifold and let $A = \{A_\alpha\}_{\alpha \in \mathcal{A}}$ be a partition of M so that $M = \coprod_\alpha A_\alpha$ with each A_α being path-connected. If there exists a cover $\mathcal{U}$ of M by open sets $\{U\}_{U \in \mathcal{U}}$ with each U homeomorphic to $\mathbb{R}^n$ (or $\mathbb{R}^n_+$) so that

$\varnothing \neq A_\alpha \cap U$ is a parallel translation of the hyperplane $\mathbb{R}^{n-c}$ in $\mathbb{R}^n$, then A is called a foliation of M of codimension c with $0 < c < n$.

For $n = 2$, we have a surface and there is an isothermal orientation preserving coordinates for the pairing $< , >$, x^1, x^2 so that $g_{ij} = \rho\delta_{ij}$. So $\phi = (x^1, x^2) : U \to V \subseteq \mathbb{R}^2$ is conformal which preserves $d\phi(S^1) = S^1$.

Now, we might ask ourselves what might be the Laplacian in these coordinates on $\Omega^0(M)$, assuming $dx^1 \perp dx^2$ and $||dx^1|| = ||dx^2||$? The $*$-operator gives us $*dx^1 = dx^2$, $*dx^2 = -dx^1$ and $\Delta = \delta d + d\delta = \delta d$. So

$$\begin{aligned} \delta d(f) &= \delta\left(f_{x^1}dx^1 + f_{x^2}dx^2\right) && (14)\\ &= (-1)^{2(p+1)+1} * d * \left(f_{x^1}dx^1 + f_{x^2}dx^2\right) && (15)\\ &= - * d\left(f_{x^1}dx^2 - f_{x^2}dx^1\right) && (16)\\ &= - * \left(f_{x^1x^1}dx^1 \wedge dx^2 + f_{x^2x^2}dx^1 \wedge dx^2\right) && (17)\\ &= -h(x^1, x^2)\left(f_{x^1x^1} + f_{x^2x^2}\right). && (18) \end{aligned}$$

Since $h(x^1, x^2)dx^1 \wedge dx^2 = \nu$, $\Delta f = 0 \Leftrightarrow f_{x^1x^1} + f_{x^2x^2} = 0$.

Note that $*\Delta = \Delta*$, $\delta\Delta = \Delta\delta$, and $\Delta d = d\Delta$.

From this point on, let's assume that M is a compact manifold and ν is a volume form. Define $< , >$ on $\Omega^p(M)$. Let $\mu, \eta \in \Omega^p(M)$. Then

$$< \mu, \eta >= \int_M \mu \wedge *\eta = \int_M \langle \mu_m, \eta_m \rangle \, \nu.$$

The pairing is bilinear which one can easily check and it is positive definite:

$$\langle \mu, \mu \rangle = \int_M \mu \wedge *\mu = \int_M < \mu_m, \mu_m > \nu \geq 0$$

since $< , >_m$ is an inner product on $\wedge^p(T_m^*M)$. So for all $m \in M$, $< \mu_m, \mu_m > \geq 0$, and $= 0 \Leftrightarrow \mu_m = 0$. So we define $||\mu|| = \sqrt{< \mu, \mu >}$

Proposition. For compact manifolds, δ is the adjoint of d:

$$< d\omega, \eta >=< \omega, \delta\eta >$$

Proof. Without loss of generality, let $\omega \in \Omega^{p-1}(M)$, $\eta \in \Omega^p(M)$. We have $d(\omega \wedge *\eta) = d\omega \wedge *\eta + (-1)^{p-1}\omega \wedge d(*\eta) = d\omega \wedge *\eta - \omega \wedge *\delta\eta$. By Stoke's theorem, $0 = \int_M d(\omega \wedge *\eta) = \int_M d\omega \wedge *\eta - \int_M \omega \wedge *\delta\eta =< d\omega, \eta > - < \omega, \delta\eta >$. ☒

Corollary. Δ is self-adjoint: $< \Delta\omega, \eta >=< \omega, \Delta\eta >$.

Proof.

$$\begin{aligned} < d\delta\omega + \delta d\omega, \eta > &=< d\delta\omega, \eta > + < \delta d\omega, \eta > && (19)\\ &=< \delta\omega, \delta\eta > + < d\omega, d\eta > && (20)\\ &=< w, d\delta\eta > + < \omega, \delta d\eta > && (21)\\ &=< \omega, \Delta\eta > \qquad \boxtimes && (22) \end{aligned}$$

Proposition. $\Delta\omega = 0 \Leftrightarrow d\omega = 0$ and $\delta\omega = 0$.

Proof. ($\Leftarrow$) Obvious.

($\Rightarrow$) Suppose $\Delta\omega = 0$. Then $0 = < \Delta\omega, \omega > = < d\delta\omega, \omega > + < \delta d\omega, \omega > = < \delta\omega, \delta\omega > + < d\omega, d\omega > = ||\delta\omega||^2 + ||d\omega||^2$ So $||\delta\omega|| = 0$ and $||d\omega|| = 0$ imply $\delta\omega = 0$ and $d\omega = 0$. ⊠

Corollary. Harmonic functions $f \in \mathcal{H}^0(M)$ on a compact manifold M are locally constants.

Corollary. We have an injective map $\mathcal{H}^p(M) \hookrightarrow H^p_{dR}(M)$.

Proof. Let $\mu \in \mathcal{H}^p(M)$ with $[\mu] = 0$. Then $\mu = d\omega$ for some ω, which implies $||\mu||^2 =< \mu, \mu >=< d\omega, \mu >=< \omega, \delta\mu >= 0$. So $\mu = 0$. ⊠

Corollary. Harmonic forms $\mathcal{H}^p(M)$ on a compact manifold M is finite dimensional. Moreover, singular, simplicial, de Rham, and Čech cohomology with real coefficients are all finite dimensional.

Example. Let $\omega \in \Omega^1(M)$ with $d\omega = 0$. On an open contractible set $U \subseteq M$, $H^1_{dR}(U) = 0$ implies we can take $w|_U = df$ so that $f(m) = \int_{m_0}^{m} \omega$, $df = \omega$ with $m_0 \in U$.

Whenever $\omega \neq 0$, $\ker(\omega)$ is a distribution and $\omega \neq 0$ implies $df \neq 0$ and thus f is a submersion. So $f^{-1}(c)$ is a submanifold with $T_m(f^{-1}(c)) = \ker(df_m) = \omega_m$.

For ω closed, $\ker\omega$ is completely integrable.

Now suppose $\int_{\cdot} \omega : \pi_1(M, m_0) \twoheadrightarrow < c > \rightarrow \mathbb{R}$ is a homomorphism onto a cyclic image $< c >$ for some c. We can then define $F := \int_{m_0}^{\cdot} \omega : M \rightarrow \mathbb{R}/ < c > \cong S^1$ that takes $m \mapsto \int_{m_0}^{m} \omega$ so that $F^*(dt) = \omega$.

Suppose $\omega \in \mathcal{H}^1(M)$ and $\int_{\cdot} \omega : \pi_1(M, m_0) \twoheadrightarrow < c > \rightarrow \mathbb{R}$. Let $F : M \rightarrow \mathbb{R}/ < c >$ and $\pi : \mathbb{R} \rightarrow \mathbb{R}/ < c >$. Locally $\omega = dF (= F^*(dt))$. Since U is an open contractible subset of M, we have a lift $\tilde{F} : U \rightarrow \mathbb{R}$ so that $F = \pi \circ \tilde{F}$. Since ω is exact, $d\omega = 0$ and $\delta\omega = 0$. So $\Delta\tilde{F} = (\cancel{d}\ \cancel{\delta} + \delta d)\tilde{F} = \delta d\tilde{F} = \delta\omega = 0$. So $\tilde{F}$ is a harmonic function, and F is locally harmonic.

If $k = \dim(H^1_{dR}(M)) = \mathcal{H}^1(M)$ and $\omega^1, ..., \omega^k$ are basis for $\mathcal{H}^1(M)$, then $\left(\int_{\cdot} \omega^1, ..., \int_{\cdot} \omega^k\right) : \pi_1(M, m_0) \twoheadrightarrow \Lambda \rightarrow \mathbb{R}^k$ where $\Lambda \subseteq \mathbb{R}^k$ is a discrete lattice with $\mathbb{R}^k/\Lambda = \mathbb{T}^k$. So $F = \left(\int_{m_0} \omega^1, ..., \int_{m_0} \omega^k\right) : M \rightarrow \mathbb{T}^k$ is a locally harmonic k-tuple.

Note that on $\mathbb{T}^k$, $dx^1 \wedge ... \wedge dx^k$ form a basis for $\mathcal{H}^1(\mathbb{T}^k)$ with $F^*(dx^j) = \omega^j$. More generally, $F^*(t_j dx^j)$ is a harmonic representative of de Rham cohomology class.

Hodge Decomposition Theorem. $\mathcal{H}^p(M)$ is finite dimensional and $\Omega^p(M)$ can be decomposed as an orthogonal direct sum $\Omega^p(M) = \Delta(\Omega^p(M)) \bigoplus \mathcal{H}^p(M)$ $= (d\delta(\Omega^p(M))) \bigoplus \delta d(\Omega^p(M)) \bigoplus \mathcal{H}^p(M) = d(\Omega^{p-1}(M)) \bigoplus \delta(\Omega^{p+1}(M)) \bigoplus \mathcal{H}^p(M)$. In particular, $\Delta\omega = \eta$ has a unique solution if $\eta \perp \mathcal{H}^p(M)$.

It is an exercise to show that the above equalities hold true. Hint: use properties discussed earlier.

Now let's show that $\Delta\omega = \eta$ has a solution for $\eta \in (\mathcal{H}^p(M))^{\perp}$.

If $\Delta\omega = \eta$ then for $\mu \in \mathcal{H}^p(M)$, $< \eta, \mu > = < \Delta\omega, \mu > = < \omega, \Delta\mu > = < \omega, 0 > = 0$. So $\eta \in (\mathcal{H}^p(M))^{\perp}$ and the solution ω gives us a bounded linear functional on $\Omega^p(M)$, $l_\omega : \Omega^p(M) \to \mathbb{R}$ so that for all $\mu \in \Omega^p(M)$, $l_\omega(\mu) =< \omega, \mu >$ and $l_\omega(\Delta\mu) =< \omega, \Delta\mu >=< \Delta\omega, \mu >=< \eta, \mu >$.

By Schwarz inequality, $|l_\omega(\mu)| = | < \omega, \mu > | \leq ||\omega||||\mu|| = c||\mu||$, where $c \in \mathbb{R}$. So l_ω is a bounded linear functional.

We say a bounded linear functional $l : \Omega^p(M) \to \mathbb{R}$ is a weak solution to $\Delta\omega = \eta$ if $l(\Delta\mu) =< \eta, \mu >$.

Regularity Theorem. Let $\eta \in \Omega^p(M)$ and let l be a weak solution of $\Delta\omega = \eta$. Then there is an $\omega \in \Omega^p(M)$ with $l = l_\omega$.

Note that the converse holds true as well: if $l = l_\omega$, then $\Delta\omega = \eta$.

Theorem. Let $\{\mu_n\} \subseteq \Omega^p(M)$ be any sequence with $||\mu_n|| \leq c$ and $||\Delta\mu_n|| \leq c$ for all n. Then there is a Cauchy subsequence $\{\mu_{n_j}\}$ in $\Omega^p(M)$.

Recall the **Hodge Decomposition Theorem**. $\mathcal{H}^p(M)$ is finite dimensional and $\Omega^p(M)$ decomposes as a orthogonal direct sum

$$\Omega^p(M) = \Delta(\Omega^p(M)) \bigoplus \mathcal{H}^p(M).$$

Equivalently, $\Delta\omega = \eta$ has a solution ω if and only if $\eta \in (\mathcal{H}^p(M))^\perp$

Proof of HDT. We need to show $(\mathcal{H}^p(M))^\perp = \Delta(\mathcal{H}^p(M))$. Since Δ is self-adjoint, let $\rho \in \mathcal{H}^p(M)$. Then for $\omega \in \Omega^p(M)$, $< \Delta\omega, \rho >=< \omega, \Delta\rho >=< \omega, 0 >= 0$. So $\Delta(\Omega^p(M)) \subseteq (\mathcal{H}^p(M))^\perp$. Now we need to show that $(\mathcal{H}^p(M))^\perp \subseteq \Delta(\Omega^p(M))$.

I claim that there exists a $c > 0$ so that for any $\mu \in (\mathcal{H}^p(M))^\perp$, $||\mu|| \leq c||\Delta\mu||$.

Proof of the claim. If not, then there exists $\{\mu_j\}$ so that $||\mu_j|| = 1$ and $||\Delta\mu_j|| \to 0$. By the above theorem, we can pass to a subsequence so we can assume that our original sequence $\{\mu_j\}$ is Cauchy. Let $l : \Omega^p(M) \to \mathbb{R}$ be defined by $l(\eta) = \lim_{j\to\infty} < \eta, \mu_j >$. It's clear that l is linear. Since $0 \leq |l(\eta)| = \lim_{j\to\infty} | < \eta, \mu_j > | \leq \lim_{j\to\infty} ||\eta||||\mu_j|| = ||\eta||$, the map l is also bounded. Moreover by Schwarz, $0 \leq |l(\Delta\eta)| = \lim_{j\to\infty} | < \Delta\eta, \mu_j > | = \lim_{j\to\infty} | < \eta, \Delta\mu_j > | \leq \lim_{j\to\infty} ||\eta||||\Delta\mu_j|| = 0$. So l is a weak solution to $\Delta\omega = 0$. By regularity theorem there is a form $\omega \in \Omega^p(M)$ with $l = l_\omega$. In other words, $l_\omega =< \cdot, \omega >$ is a solution to the equation, and $l_\omega(\Delta\eta) = 0$ which implies $\Delta\omega = 0$. If we also have $l_{\mu_j} \to l_\omega$, then μ_j converges to ω. So $\{\mu_j\} \subseteq (\mathcal{H}^p(M))^\perp$, $||\mu_j|| = 1$, and $\lim_{j\to\infty} \mu_j = \omega \in \mathcal{H}^p(M)$. But $(\mathcal{H}^p(M))^\perp$ and $\mathcal{H}^p(M)$ are closed. This is a contradiction. This proves the claim.

Now we need to show that $(\mathcal{H}^p(M))^\perp \subseteq \Delta(\Omega^p(M))$, so let $\eta \in (\mathcal{H}^p(M))^\perp$. Define a linear functional $l : \Delta(\Omega^p(M)) \to \mathbb{R}$ where $(\Delta\mu) =< \eta, \mu >$. Show that the linear functional is well-defined: if we have μ, μ' with $\Delta\mu = \Delta\mu'$, then we need to show $< \eta, \mu >=< \eta, \mu' >$. Since $\Delta\mu = \Delta\mu'$, $\Delta(\mu - \mu') = 0$. Then $\mu - \mu' \in \mathcal{H}^p(M)$ is harmonic. So $< \eta, \mu > - < \eta, \mu' >=< \eta, \mu - \mu' >= 0$ since $\eta \in (\mathcal{H}^p(M))^\perp$. So $< \eta, \mu >=< \eta, \mu' >$ and the functional is well-defined. Also by Schwarz inequality, $|l(\Delta\mu)| = | < \eta, \mu > | \leq ||\eta||||\mu|| \leq ||\eta|| \cdot c||\Delta\mu|| = c'||\Delta\mu||$. So the functional is bounded.

Hahn-Banach theorem tells us that l extends to a bounded linear functional on $l : \Omega^p(M) \to \mathbb{R}$ and for all $\mu \in \Omega^p(M)$, $l(\Delta\mu) =< \eta, \mu >$. So l is a weak solution. By Regularity theorem, $l = l_\omega$ and $\Delta\omega = \eta$. So $(\mathcal{H}^p(M))^\perp \subseteq \Delta(\Omega^p(M))$. This proves the other direction of the Hodge Decomposition theorem. $\boxtimes$

Recall that $\mathcal{H}^p(M) \hookrightarrow H^p_{dR}(M)$ is injective.

Theorem. We have an isomorphism $\mathcal{H}^p(M) \cong H^p_{dR}(M)$

The key idea of the proof is that we invert the Laplacian by defining Green's operator $G : \Omega^p(M) \to (\mathcal{H}^p(M))^\perp = \Delta(\Omega^p(M))$ by we project to the complement of the kernel and then invert; i.e., let $\Pi = I - H : \Omega^p(M) \to (\mathcal{H}^p(M))^\perp$ be the orthogonal projection map, $\Delta : (\mathcal{H}^p(M))^\perp \subseteq \Omega^p(M) \to (\mathcal{H}^p(M))^\perp$ is an isomorphism, and $G : \Omega^p(M) \to (\mathcal{H}^p(M))^\perp$ so that $\Delta \circ G = \Pi$.

Lemma. Any operator T that commutes with Δ commutes with G. In particular, $*G = G*$, $\delta G = G\delta$, $dG = Gd$, and $\Delta G = G\Delta$.

Proof of Theorem. Let $[\omega] \in H^p_{dR}(M)$. Then $d\omega = 0$. We need to find harmonic representative (form) for ω. So it suffices to show $[\mathcal{H}(\omega)] = [\omega]$. By Hodge Decomposition theorem, $w = \Delta(G\omega) + \mathcal{H}(\omega) = d\delta G\omega + \delta(dG)\omega + \mathcal{H}(\omega)$ $= d(\delta G\omega) + \delta G(d\omega) + \mathcal{H}(\omega) = d(\delta G\omega) + \delta G(0) + \mathcal{H}(\omega) = d(\delta G\omega) + \mathcal{H}(\omega)$. Thus $[\omega] = [\mathcal{H}(\omega)]$. $\boxtimes$

The next proposition tells us that that the harmonic representative is the best representative with respect to the metric.

Proposition. The harmonic representative of $[\omega] \in H^p_{dR}(M)$ is the unique norm minimizer in the class.

Proof. Let $\omega \in \mathcal{H}^p(M)$ be the harmonic representative and let $\eta \in \Omega^{p-1}(M)$. We need to check that $||\omega|| \le ||\omega + d\eta||$, and with equality if and only if $d\eta = 0$. So

$$\begin{aligned}
||\omega + d\eta||^2 &= ||\omega||^2 + ||d\eta||^2 + 2 < \omega, d\eta > && (23)\\
&= ||\omega||^2 + ||d\eta||^2 + 2 < \delta\omega, \eta > && (24)\\
&= ||\omega||^2 + ||d\eta||^2 + 2 < 0, \eta > && (25)\\
&= ||\omega||^2 + ||d\eta||^2 && (26)\\
&= ||\omega||^2 \text{ if and only if } d\eta = 0. && (27)
\end{aligned}$$

We do have that $d\eta = 0$ because $||\cdot||$ is a norm. $\boxtimes$