Mee Seong Im Introduction to Logic and Proofs Monday July 28, 2008

Problem 13.3. [page 268]. Give a counterexample to the following false statement, and add one word to correct it (other than "not").

"Every bounded sequence of real numbers converges."

I would change the above statment to: every bounded monotone sequence of real numbers converges.

Problem 13.11. Suppose that $< a >$ and $< b >$ converge.

a) If $\lim a_n < \lim b_n$, then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n < b_n$.

b) If $\lim a_n \leq \lim b_n$, then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \leq b_n$.

a) True. **Proof for a contradiction**. Suppose for all $N \in \mathbb{N}$ with $n \geq N$, $a_n \geq b_n$. Then since $< a >$ and $< b >$ converge, $\lim a_n$ and $\lim b_n$ exist. So $\lim_n a_n \geq \lim_n b_n$. This is a contradiction. $\boxtimes$

b) False. Take $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$. Then $\lim a_n = 0 \leq \lim b_n = 0$. However if we take N to equal 2, then for all $n \geq 2$ we have $a_n > b_n$.

Problem 13.25. Use the definition of limit to prove that $\lim \sqrt{1 + n^{-1}} = 1$.

Some analysis. Since $n \geq 1$, we have that $1 + \frac{1}{n} \geq 1$ since $\frac{1}{n}$ is a positive number. Applying the square root to both sides, we get

$$\sqrt{1 + \frac{1}{n}} \geq 1 \ \Rightarrow \ \sqrt{1 + \frac{1}{n}} - 1 \geq 0.$$

We want to show that $\lim \sqrt{1 + \frac{1}{n}} - 1 = 0$. This means that we need to show that $\forall \epsilon > 0 \ \exists N = N(\epsilon) > 0$ such that $\forall n \geq N$, $|g(n) - 0| < \epsilon$ where $g(n) = \sqrt{1 + \frac{1}{n}} - 1$. So here is the proof. [Note that the above was an analysis of trying to figure out what's going on. The above is not a proof.]

Proof. Let $\epsilon > 0$ be fixed. We need to find some $N = N(\epsilon) \gg 0$ ($N(\epsilon)$ means *N depends on ϵ*) so that for every $n \geq N$ we can conclude that $|g(n)| < \epsilon$.

Since $g(n) \geq 0$, $|g(n)| = g(n)$. That is, we can ignore the absolute value sign, look for this large natural number N, and show that $\forall n \geq N \ g(n) < \epsilon$.

Since we're trying to make $\sqrt{1 + \frac{1}{n}} - 1 < \epsilon$, solve for n and then take N to equal this n:

$$\sqrt{1+\frac{1}{n}} < \epsilon + 1 \tag{1}$$

$$1+\frac{1}{n} < (\epsilon+1)^2 \tag{2}$$

$$\frac{1}{n} < (\epsilon+1)^2 - 1 \tag{3}$$

$$\frac{1}{(\epsilon+1)^2-1} < n. \tag{4}$$

Note that $(\epsilon+1)^2 - 1 \neq 0$, so the denominator is well-defined. [The denominator equals zero if and only if $\epsilon = 0$. But then this would be a contradiction.]

Now take $\mathbf{N} > \frac{\mathbf{1}}{(\epsilon+\mathbf{1})^{\mathbf{2}}-\mathbf{1}}$.

We will now check if this N works by doing the following: for any $n \geq N$,

$$\begin{aligned}
|g(n)| &= \sqrt{1+\frac{1}{n}} - 1 && && (5)\\
&\leq \sqrt{1+\frac{1}{N}} - 1 && \text{since } \frac{1}{N} \geq \frac{1}{n} && (6)\\
&< \sqrt{1+((\epsilon+1)^2-1)} - 1 && \text{since } N > \frac{1}{(\epsilon+1)^2-1} && (7)\\
&= \epsilon + 1 - 1 = \epsilon. && && (8)
\end{aligned}$$

Thus our $N \gg 0$ was a good choice to prove that for all $n \geq N$ $|g(n)| < \epsilon$. ☒

Problem 13.28. Suppose that $x_n \to 0$ and that $|y_n| \leq 1$ for $n \in \mathbb{N}$. Find the flaw in the following computation for $\lim(x_n y_n)$, and give a valid proof that $\lim(x_n y_n) = 0$:

$$\lim(x_n y_n) = \lim(x_n)\lim(y_n) = 0 \cdot \lim(y_n) = 0.$$

The flaw in the above computation is at the first equality. Assuming only the above conditions, it is possible that $\lim(x_n y_n) \neq \lim(x_n)\lim(y_n)$ [example: let $y_n = (-1)^n$. Then $\lim(y_n)$ doesn't exist. If it were to exist, then every subsequence of $\{y_n\}_{n=1}^{\infty}$ converges to the **same** limit. Consider the following two subsequences: $y_{2n} = 1 \to 1$ whereas $y_{2n+1} = -1 \to -1$. Since $\{y_{2n}\}_{n=1}^{\infty}$ and $\{y_{2n+1}\}_{n=1}^{\infty}$ converge to different limits, the original sequence $\{y_n\}_{n=1}^{\infty}$ does NOT have a limit. So it doesn't make sense to talk about $\lim(y_n)$].

Now to prove that $\lim_{n\to\infty}(x_n y_n) = 0$, we need to show that

$$\forall \epsilon > 0 \, \exists N = N(\epsilon) > 0 \text{ s.t. } \forall n \geq N \; |x_n y_n - 0| < \epsilon.$$

Proof. Let $\epsilon > 0$ be fixed. Since $|y_n| \leq 1$ for all $n \in \mathbb{N}$, we have $0 \leq |x_n y_n| = |x_n||y_n| \leq |x_n|$. By assumption, $x_n \to 0$ [i.e., $\lim x_n = 0$]. This means we can *assume* the following as well: given any $\eta > 0$ (so this is true even for our fixed positive ϵ!), there is an $M = M(\eta) > 0$ so that $\forall n \geq M$ $|x_n| < \eta$. So by letting $\eta = \epsilon$, we still need to find N right? So take $N = M$ [the assumption $x_n \to 0$ did all the work for us!]. Then for all $n \geq N$, $|x_n y_n - 0| \leq |x_n| < \epsilon$, which is what we wanted to show. ⊠

Problem 13.29. Let $x_n = (1 + n)/(1 + 2n)$. Prove that $\lim_{n\to\infty} x_n$ exists by using Monotone Convergence. Prove that $\lim_{n\to\infty} x_n = 1/2$ by using the definition of limit.

To use Monotone Convergence Theorem [MCT, Thm 13.16, page 261], we need to show two things: 1. show that x_n is monotone decreasing or monotone increasing, and 2. show that x_n is bounded below (if monotone decreasing) or bounded above (if monotone increasing). Then by MCT, we conclude that x_n converges to a limit as $n \to \infty$.

Here, we're going to show that x_n is *monotone decreasing* and x_n is *bounded below*. How do we know that x_n is not monotone increasing? Just plug-in and compare x_1 and x_2: $x_1 = 2/3 \geq 3/5 = x_2$. Since the problem asks us to use MCT, we know x_n is a monotone function. Since $x_1 \geq x_2$, we know further that x_n is monotone decreasing.

Proof. Let $n \in \mathbb{N}$. Then comparing two consecutive terms,

$$\frac{1+n}{1+2n} \geq \frac{1+(n+1)}{1+2(n+1)} \tag{9}$$

$$\Leftrightarrow (1+n)(3+2n) \geq (n+2)(1+2n) \tag{10}$$

$$\Leftrightarrow 3+3n+2n+2n^2 \geq n+2+2n^2+4n \tag{11}$$

$$\Leftrightarrow 5n+3 \geq 5n+2 \tag{12}$$

$$\Leftrightarrow 3 \geq 2 \tag{13}$$

$$\Leftrightarrow 1 \geq 0. \tag{14}$$

It is true that $1 \geq 0$, so $\frac{1+n}{1+2n} \geq \frac{1+(n+1)}{1+2(n+1)}$ holds as well. Since n was an arbitrary natural number, this holds for all $n \in \mathbb{N}$. So for all $n \in \mathbb{N}$ $x_n \geq x_{n+1}$.

Now show that x_n is bounded below. Since $n > 0$, $1+n > 0$ and $1+2n > 0$. Thus $\frac{1+n}{1+2n} \geq 0$. By MCT, x_n has a limit. ⊠

We could have said that $\frac{1+n}{1+2n} > 0$, but it is not wrong to write $\geq$ sign instead of the $>$ sign. You could also show that x_n is bounded below by some positive number, which better approximates the limit of x_n but also requires more work to be done here. Since the theorem says all you need to do is find **some** lower bound, above analysis is the shortest/fastest way to do so.

Proof. Let $\epsilon > 0$. In order to prove $\lim_{n\to\infty} x_n = 1/2$, we need to find $N = N(\epsilon) > 0$ so that for all $n \geq N$ $|x_n - 1/2| < \epsilon$. Since

$$\begin{aligned} |x_n - 1/2| &= \left|\frac{1+n}{1+2n} - \frac{1}{2}\right| && (15) \\ &= \left|\frac{2+2n-1-2n}{2+4n}\right| && (16) \\ &= \left|\frac{1}{2+4n}\right| = \frac{1}{2+4n} < \epsilon, && (17) \end{aligned}$$

I claim that we need to take $N > \frac{1-2\epsilon}{4\epsilon}$. [If you want to know how I got this, just solve for n and take N to be this (or greater than or equal to this) n.]

Then for all $n \geq N$,

$$|x_n - 1/2| = \frac{1}{2+4n} \leq \frac{1}{2+4N} < \epsilon \tag{18}$$

because since $N > \frac{1-2\epsilon}{4\epsilon}$,

$$\begin{aligned} &4N > \frac{1-2\epsilon}{\epsilon} && (19) \\ &\Rightarrow 2 + 4N > \frac{1-2\epsilon}{\epsilon} + 2\frac{\epsilon}{\epsilon} = \frac{1}{\epsilon} && (20) \\ &\Rightarrow \epsilon > \frac{1}{2+4N}. \quad \boxtimes && (21) \end{aligned}$$