Mee Seong Im   Introduction to Logic and Proofs   Monday July 28, 2008

Problem 14.3. [page 287]. For each condition below, give examples of sequences $< a >$ and $< b >$ such that $\lim a_n = 0$, $\lim b_n$ does not exist, and the specified condition holds.

a) $\lim(a_n b_n) = 0$.

b) $\lim(a_n b_n) = 1$.

c) $\lim(a_n b_n)$ does not exist.

a) Take $a_n = \frac{1}{n}$ and $b_n = (-1)^n$.

b) Take $a_n = (-1)^n \frac{1}{n}$ and $b_n = (-1)^n n$.

c) Take $a_n = \frac{1}{n}$ and $b_n = (-1)^n n^2$. Then $\lim(a_n b_n) = \lim(-1)^n n$ does not exist.

Problem 14.5. Find a counterexample to the following false statement.

"If $a_n < b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges."

This is false. Here is the counterexample: for all n take $a_n = -1$ and $b_n = 0$. Then $a_n < b_n$ for every n and $\sum b_n = \sum 0 = 0 < \infty$. However, $\sum a_n = \sum -1 = -\infty$. So $\sum a_n$ diverges.

QUESTION: What would you add to make the above statement true?

For this exercise, determine whether the statement is true or false. If true, provide a proof; if false, provide a counterexample.

Problem 14.8. Let $< x >$ be a sequence of real numbers.

a) If $< x >$ is unbounded, then $< x >$ has no limit.

b) If $< x >$ is not monotone, then $< x >$ has no limit.

a) True. **Proof for a Contradiction**. Suppose $< x >$ has a limit. Then $\lim_{n\to\infty} x_n = L < \infty$. Then given any $\epsilon > 0$, there exists $N = N(\epsilon) > 0$ such that for all $n \geq N$ $|x_n - L| < \epsilon$. This implies that $< x >$ is bounded. Contradiction. ⊠

b) False. Let $x_n = (-1)^n \frac{1}{n}$. Then $< x_n >$ is not monotone but $\lim x_n = 0$.

Problem 14.15. Suppose that $b \leq L + \epsilon$ for all $\epsilon > 0$. Prove that $b \leq L$.