Problem 4.11 [page 95]. Explain why multiplication by 2 defines a bijection from $\mathbb{R}$ to $\mathbb{R}$ but not from $\mathbb{Z}$ to $\mathbb{Z}$.

Argument. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = 2x$ for all $x \in \mathbb{R}$.

Show that f is *injective*: assume $f(x) = 0$. This implies that $2x = 0$. Since $2 \neq 0$, we can divide both sides by 2 to get $x = 0$. So f is injective.

Now show that f is *surjective*: let $y \in \mathbb{R}$. We need to find some x so that f takes x to y, i.e., $f(x) = y$. So just suppose that we found this x so that $f(x) = y$. Since we know that $f(x) = 2x$, $2x = y$. So if we take x to be $x = y/2$, then this element $y/2$ will be mapped to y under f.

Let's check: $f(x) = f(y/2) = 2(y/2) = y$. So f is surjective.

Thus, $f : \mathbb{R} \to \mathbb{R}$ where $f = 2$ is bijective.

Now let $f : \mathbb{Z} \to \mathbb{Z}$ be the function $f(n) = 2n$ for all $n \in \mathbb{Z}$.

It suffices to show that f is not surjective. This will imply that f is not bijective.

Proof. For a contradiction, suppose f is surjective. Consider $1 \in \mathbb{Z}$. Since f is surjective, there is $n \in \mathbb{Z}$ so that $f(n) = 1$. Since $f(n) = 2n$, $2n = 1$. Let's divide both sides by 2 because 2 is a nonzero number. We see that $n = 1/2$, which is not an integer. This is a contradiction because there is no $n \in \mathbb{Z}$ which is mapped to 1. Thus f is not surjective.

Hence $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 2n$ for all $n \in \mathbb{Z}$ is not a bijective function.

[I wrote a lot more than I needed but they have been written to make sure that you understand and can justify every step in the argument.]