Problem 4.20 [page 96]. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (ax - by, bx + ay)$, where a, b are numbers with $a^2 + b^2 \neq 0$.

a.) Prove that f is a bijection.

b.) Find a formula for f^{-1}.

c.) Give a geometric interpretation of f for a case $a^2 + b^2 = 1$. (Describe the effect f has on geometric figures in the plane.)

a.) Recalling from linear algebra, we can write $f(x, y)$ as

$$f(x,y) = f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

Let

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Then f is invertible if and only if A is invertible.
Since $\det(A) = a^2 + b^2 \neq 0$, A is invertible. So

$$A^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

and

$$f^{-1}\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = A^{-1}\begin{bmatrix} s \\ t \end{bmatrix} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} \tag{1}$$

$$= \left[\frac{as+bt}{a^2+b^2}, \frac{-bs+at}{a^2+b^2}\right] = \begin{bmatrix} x \\ y \end{bmatrix} \tag{2}$$

Show f *injective*: suppose $f(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$ where $\mathbf{x} = (x, y)$. Since A invertible, multiply both sides on the left by A^{-1} to obtain $\mathbf{x} = \mathbf{0}$. So f is injective.

Show f *surjective*: since f is injective and f^{-1} exists (and f^{-1} also being injective by similar argument), f is surjective.

Thus f is bijective.

b.) The formula for f^{-1} is

$$f^{-1}\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = A^{-1}\begin{bmatrix} s \\ t \end{bmatrix} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix}.$$

c.) Now let us assume that $a^2 + b^2 = 1$. Given any $\mathbf{x} = (x, y)$ and $\mathbf{y} = (s, t)$ if I show that f is a distance preserving map, i.e., $d(f(\mathbf{x}), f(\mathbf{y})) = d(\mathbf{x}, \mathbf{y})$, then f is an isometry. So

$$d\left(f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right), f\left(\begin{bmatrix} s \\ t \end{bmatrix}\right)\right) = \left|\left|f\begin{bmatrix} x \\ y \end{bmatrix} - f\begin{bmatrix} s \\ t \end{bmatrix}\right|\right| \tag{3}$$

$$= \left|\left|A\begin{bmatrix} x \\ y \end{bmatrix} - A\begin{bmatrix} s \\ t \end{bmatrix}\right|\right| \tag{4}$$

$$= ||A|| \cdot \left|\left|\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} s \\ t \end{bmatrix}\right|\right| \tag{5}$$

$$= \det(A) \cdot \left|\left|\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} s \\ t \end{bmatrix}\right|\right| \tag{6}$$

$$= (a^2 + b^2) \cdot d\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} s \\ t \end{bmatrix}\right) \tag{7}$$

$$= d\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} s \\ t \end{bmatrix}\right) \tag{8}$$

Thus if we assume that $a^2 + b^2 = 1$, then f is an isometry.

Challenge: Now can you figure out whether f is orientation-preserving or orientation-reversing? Are there any lines of symmetry and if so, where?