

# The Hodge Decomposition Theorem

(Adapted from Warners book  
“Foundations of Differential Manifolds and Lie Groups”)

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Throughout this write up, we assume  $M$  is a compact oriented Riemannian manifold of dimension  $n$ . The Hodge decomposition theorem deals with the question of solvability of the following linear partial differential equation

$$\Delta\omega = \alpha \tag{1}$$

in the space of all smooth  $p$ -forms  $E^p(M)$  where  $\alpha \in E^p(M)$ . In general, no solution is possible. The Hodge decomposition theorem provides us with a necessary and sufficient condition for the solvability. Namely, it states that if  $\alpha \in E^p(M)$ , then (1) has a solution  $\omega \in E^p(M)$  if and only if  $\alpha$  is orthogonal (in some suitable inner product on  $E^p(M)$ ) to the kernel of the operator  $\Delta : E^p(M) \rightarrow E^p(M)$ . Such results should be familiar from elementary linear algebra and functional analysis. There, we have a result known as the Fredholm alternative, which states that if  $A$  is a bounded linear operator on a Hilbert space  $H$ , then  $H = \bar{R}(A) \oplus N(A^*)$  where  $R(A)$  denotes the range of  $A$  and  $N(A)$  denotes the null space (or kernel) of  $A$ . In particular, this says that if  $A$  has closed range, then the equation  $Ax = b$  has a solution if and only if  $b \in (N(A))^\perp$ , i.e. if and only if  $\langle b, h \rangle = 0$  for all  $h \in N(A)$ . In the case when  $H = \mathbb{R}^n$ , this is essentially the elementary result that  $Ax = b$  has a solution for all  $b \in H$  if and only if  $Ax = 0$  has only the trivial solution.

In the context of differential equations, in particular for the Laplace equation, the Fredholm alternative takes the following form:

**Theorem 1.** *Let  $U \subset \mathbb{R}^n$  be a bounded open subset. Given  $f \in L^2(U)$ , the boundary value problem*

$$\begin{aligned} \Delta u &= f \text{ in } U \\ u|_{\partial U} &= 0 \text{ on } \partial U \end{aligned}$$

*has a (weak) solution if and only if*

$$\langle f, v \rangle = 0 \text{ for all } v \in \text{Ker}(\Delta).$$

The proof of this theorem relies on a version of the Fredholm alternative for compact operators, and can be found in many elementary books on linear PDE's (for example, see [1]).

The purpose of this write up is to extend the above familiar results to smooth  $p$ -forms on compact oriented Riemannian manifolds. We begin by defining a suitable generalization of the Laplace operator on such manifolds and defining an appropriate inner product on  $E^p(M)$  as to make it a Hilbert space.

## 1 The Laplace-Beltrami Operator

We begin by recalling the basic properties of the Hodge  $*$  operator on  $M$ . This is a linear operator  $*$  :  $\bigwedge^p \rightarrow \bigwedge^{n-p}$  which satisfies  $** = (-1)^{p(n-p)}$ . Using this, we define a linear operator  $\delta$  :  $\bigwedge^p \rightarrow \bigwedge^{p-1}$  by

$$\delta = (-1)^{n(p+1)+1} * d*$$

Notice that on  $\bigwedge^0$ ,  $\delta$  is just the zero functional. The Laplace-Beltrami operator  $\Delta$  :  $E^p(M) \rightarrow E^p(M)$  for each  $0 \leq p \leq n$  is defined by

$$\Delta = \delta d + d\delta.$$

**Example 1.** *This example proves the Laplace-Beltrami operator is really a natural extension of the standard Laplace operator familiar from elementary calculus. Let  $u \in C^\infty(\mathbb{R}^2)$ . Then an easy calculation shows*

$$\begin{aligned} (\delta d + d\delta)u &= (\delta d)u \\ &= (-1) * d * (u_x dx + u_y dy) \\ &= (-1) * d(u_x dy - u_y dx) \\ &= (-1) * (u_x x dx \wedge dy - u_y y dy \wedge dx) \\ &= (-1)(u_x x + u_y y). \end{aligned}$$

Thus,  $-\Delta = \partial_x^2 + \partial_y^2$  on  $E^0(\mathbb{R}^2)$ .

We now define an inner product on the space  $E^p(M)$  which we then extend to  $\sum_{p=0}^n E^p(M)$  by linearity: given  $\alpha, \beta \in E^p(M)$  we define

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge * \beta.$$

This inner product is natural in the sense that  $\delta$  is the formal adjoint of  $d$  on the  $(L^2)$  inner product space  $\sum_{p=0}^n E^p(M)$ . Indeed, notice that given  $\alpha \in E^{p-1}(M)$  and  $\beta \in E^p(M)$ .

$$d(\alpha \wedge * \beta) = d\alpha \wedge * \beta - \alpha \wedge * \delta \beta$$

and hence Stokes theorem implies that

$$0 = \int_M (d\alpha \wedge * \beta - \alpha \wedge * \delta \beta) = \langle d\alpha, \beta \rangle - \langle \alpha, \delta \beta \rangle.$$

It follows that  $\Delta$  is (formally) self adjoint on  $\sum_{p=0}^n E^p(M)$ .

## 2 Weak Solutions

When attempting to prove the existence of a solution to a linear PDE, one could easily be bogged down with subtle questions of differentiability and the prospect of being able to interchange the order of derivatives. The concept of a weak solution allows you to dispense with these subtleties pretty much altogether to be able to prove that some solution exists in a larger space. Then, once you know there is a solution in some larger space, one can

begin more detailed analysis to prove this solution is in fact as smooth as you desired. An analogy in elementary algebra would be trying to find rational roots of polynomials. In the beginning of your investigation, it is important to know that a solution exists in a larger category, say in the real numbers. Many techniques exist to aid you in this larger category which are not available to you in the smaller class of rationals (e.g. Roll's theorem and the intermediate value theorem). Once you have established existence in the larger space of real numbers, you can try to prove that in fact the solutions are rational.

This motivates us to find a suitable "weaker" definition of what is meant by a solution to a PDE which is easier to work with and subtle questions about differentiability can essentially be ignored. To begin we look at what happens in the scalar case familiar from elementary linear PDE's. Consider the operator  $\Delta = \partial_x^2 + \partial_y^2$  on  $\mathbb{R}^2$ . Given  $g \in C^\infty(\mathbb{R}^2)$ , we consider the equation  $\Delta u = g$  and ask when do we have, say smooth, solutions to this problem. Well, assume  $u$  is some smooth solution and let  $v \in C_c^\infty(\mathbb{R}^2)$ . Then integration by parts implies

$$\int_{\mathbb{R}^2} g(x)v(x)dx = \int_{\mathbb{R}^2} u(x)\Delta v(x)dx$$

This suggests we can view the solution  $u$  of our differential equation as a linear functional on  $C_c^\infty(\mathbb{R}^2)$  by  $L(f) = \langle u, f \rangle_{L^2}$  and notice that from above, the functional  $L$  must satisfy

$$L(\Delta v) = \langle g, v \rangle_{L^2(\mathbb{R}^2)}.$$

for all  $v \in C_c^\infty(\mathbb{R}^2)$ . Clearly, this implies that every smooth solution gives rise to such a linear functional. The converse statement is not in general true. However, this can serve as our motivation for the concept of a "weak" solution to the above PDE.

**Definition 1.** We call a linear functional  $L : C_c^\infty \rightarrow \mathbb{R}$  a weak solution of the equation  $\Delta u = g$  if  $L(\Delta f) = \langle g, f \rangle_{L^2}$  for all  $f \in C_c^\infty(\mathbb{R}^2)$ .

Notice that we do not have any questions about the differentiability of  $u$  in the above definition.

We now generalize these comments to handle our case of compact Riemannian manifolds. As above, we begin by assuming  $\alpha \in E^p(M)$  is given and that  $\omega \in E^p(M)$  is a solution of  $\Delta\omega = \alpha$ . Then taking the inner product with an arbitrary  $\phi \in E^p(M)$  yields

$$\langle \Delta\omega, \phi \rangle = \langle \alpha, \phi \rangle.$$

Since  $\Delta$  is self adjoint on  $\sum_{p=0}^n E^p(M)$ , this implies that  $\langle \omega, \Delta\phi \rangle = \langle \alpha, \phi \rangle$  and hence we may view our solutions  $\omega$  as a linear functional  $L : E^p(M) \rightarrow \mathbb{R}$  defined by  $L(\delta) = \langle \omega, \delta \rangle$ . It follows that  $L(\Delta\phi) = \langle \alpha, \phi \rangle$  for all  $\phi \in E^p(M)$ . As above, this motivates the following definition.

**Definition 2.** Given  $\alpha \in E^p(M)$ , a weak solution of the equation  $\Delta\omega = \alpha$  is a linear functional  $L : E^p(M) \rightarrow \mathbb{R}$  which satisfies  $L(\Delta\phi) = \langle \alpha, \phi \rangle$  for all  $\phi \in E^p(M)$ .

By our above derivation, it is clear that an ordinary solution is a weak solution. However, it is not at all clear if the converse holds. This is actually the majority of the work when studying linear PDE theory. It is usually not difficult to establish the existence of a weak solution, but it is sometimes quite difficult to force additional regularity. In our current case, the fact that the converse holds is handled in the following theorem.

**Theorem 2** (Regularity theorem). *Let  $\alpha \in E^p(M)$ , and let  $L$  be a weak solution of  $\Delta\omega = \alpha$ . Then there exists a  $\omega \in E^p(M)$  such that  $L(\beta) = \langle \omega, \beta \rangle$  for every  $\beta \in E^p(M)$ .*

**Remark 1.** *It follows that the  $\omega$  in the regularity theorem is indeed an ordinary solution of (1). Indeed, notice that*

$$\langle \Delta\omega, \beta \rangle = \langle \omega, \Delta\beta \rangle = \langle \alpha, \beta \rangle$$

for all  $\beta \in E^p(M)$ . Hence  $\langle \Delta\omega - \alpha, \beta \rangle = 0$  for all  $\beta \in E^p(M)$  and hence  $\Delta\omega = \alpha$ .

The regularity theorem is absolutely essential when studying PDE's since it allows one to upgrade a known weak solution to a classical solution. The proof is quite complicated, needing some moderately sophisticated ideas from the theory of Sobolev spaces on compact Riemannian manifolds and some basic estimates on linear elliptic operators. Thus, we will assume this theorem holds for the meantime and will move on to our discussion of the Hodge decomposition theorem.

### 3 The Hodge Decomposition Theorem

When trying to construct solutions of a given differential equation, we have seen that it is beneficial to begin your analysis by studying the kernel of the associated differential operator. In our case, we define

$$H^p = \{\omega \in E^p(M) : \Delta\omega = 0\}$$

and say the elements of  $H^p$  are the harmonic  $p$ -forms. The Hodge decomposition theorem states that a solution of (1) exists if and only if  $\alpha \in (H^p)^\perp$  in the above defined inner product. Before we state precisely the Hodge theorem, we begin with some preliminary lemmas needed in the proof.

**Lemma 1.** *Let  $\{\alpha_n\}$  be a sequence of uniformly bounded smooth  $p$ -forms on  $M$  such that  $\{\|\Delta\alpha_n\|\}$  is uniformly bounded. Then there exists a Cauchy subsequence of  $\{\alpha_n\}$  in  $E^p(M)$ .*

**Remark 2.** *The above lemma is essentially a compactness result in the Sobolev space  $H_2(M)$ .*

**Lemma 2.** *There exists a constant  $C > 0$  such that*

$$\|\beta\| \leq C\|\Delta\beta\|$$

for all  $\beta \in (H^p)^\perp$ .

*Proof.* If not, then we can find a sequence  $\beta_n \in (H^p)^\perp$  such that  $\|\beta_n\| = 1$  for all  $n$  and  $\lim_{n \rightarrow \infty} \|\Delta\beta_n\| = 0$ . By the above lemma, we can assume, by possibly passing to a subsequence, the sequence  $\{\beta_n\}$  is Cauchy in  $E^p(M)$ . Thus, the limit  $\lim_{n \rightarrow \infty} \langle \beta_n, \psi \rangle$  exists for each  $\psi \in E^p(M)$  and hence we can define a linear functional  $F$  on  $E^p(M)$  by

$$F(\psi) := \lim_{n \rightarrow \infty} \langle \beta_n, \psi \rangle.$$

Clearly,  $F$  is a bounded linear functional and hence

$$F(\Delta\phi) = \lim_{n \rightarrow \infty} \langle \beta_n, \Delta\phi \rangle = \lim_{n \rightarrow \infty} \langle \Delta\beta_n, \phi \rangle = 0$$

for every  $\phi \in E^p(M)$ . Hence,  $F$  is a weak solution of  $\Delta\beta = 0$ . By the regularity theorem, there exists a  $\beta \in E^p(M)$  such that  $F(\psi) = \langle \beta, \psi \rangle$  for all  $\psi \in E^p(M)$  and hence  $\beta \in H^p$ . It follows that  $\beta_n \rightarrow \beta$ , and hence  $\|\beta\| = \lim_{n \rightarrow \infty} \|\beta_n\| = 1$ . But  $(H^p)^\perp$  is a closed linear subspace of  $E^p(M)$  and hence  $\beta \in (H^p)^\perp$ . This implies  $\beta = 0$ , which is a contradiction.  $\square$

**Remark 3.** *The above lemma is essentially a generalization of a Poincaré inequality from Sobolev space theory.*

We are now prepared to present a proof of the Hodge decomposition theorem.

**Theorem 3 (The Hodge Decomposition Theorem).** *For each integer  $0 \leq p \leq n$ ,  $H^p$  is finite dimensional, and we have the following orthogonal direct sum decomposition of the space  $E^p(M)$ :*

$$E^p(M) = H^p \oplus \Delta(E^p(M)).$$

**Remark 4.** *As a corollary, we see that the equation  $\Delta\omega = \alpha$  has a solution in  $E^p(M)$  if and only if  $\alpha \in (H^p)^\perp$ .*

*Proof.* The fact that  $H^p$  is finite dimensional follows directly by the above compactness lemma. Indeed, if  $H^p$  were infinite dimensional, then we could have an infinite sequence of orthonormal elements, which would then have to contain a Cauchy subsequence, which can not happen.

Since  $H^p$  is finite dimensional, we have a finite orthonormal basis, say given by  $\{w_1, \dots, w_k\}$ . Then any arbitrary  $\alpha \in E^p(M)$  can be uniquely written as

$$\alpha = \beta + \sum_{j=1}^k \langle \alpha, w_j \rangle w_j$$

where  $\beta \in (H^p)^\perp$ . Thus, we have an orthogonal direct sum decomposition

$$E^p(M) = H^p + (H^p)^\perp.$$

The theorem will follow by proving  $(H^p)^\perp = \Delta(E^p(M))$ .

Let  $\alpha \in (H^p)^\perp$  and define the linear functional  $L$  by

$$L(\Delta\phi) = \langle \alpha, \phi \rangle$$

for all  $\phi \in E^p(M)$ . Clearly  $L$  is well defined, for if  $\Delta\phi_1 = \Delta\phi_2$ , then  $\phi_1 - \phi_2 \in H^p$ , so that  $\langle \alpha, \phi_1 - \phi_2 \rangle = 0$ . To prove that  $L$  is bounded on  $\Delta(E^p)$  we define the projection  $H := E^p(M) \rightarrow H^p$ , so that  $H(\alpha)$  is the harmonic part of  $\alpha$ . Let  $\phi \in E^p(M)$  and let  $\psi = \phi - H(\phi)$ . Then using the above Poincaré type lemma we have

$$\begin{aligned} |L(\Delta\phi)| = |L(\Delta\psi)| &= |\langle \alpha, \psi \rangle| \\ &\leq \|\alpha\| \cdot \|\psi\| \\ &\leq C\|\alpha\| \cdot \|\Delta\psi\| \\ &= C\|\alpha\| \cdot \|\Delta\phi\|. \end{aligned}$$

Therefore,  $L$  is a bounded linear functional on  $\Delta(E^p)$ , and hence extends to a bounded linear functional on all of  $E^p(M)$  by the Hahn-Banach theorem. Therefore,  $L$  is a weak

solution of  $\Delta\omega = \alpha$  by definition. By the regularity theorem, there exists a  $\omega \in E^p(M)$  such that  $L(\phi) = \langle \omega, \phi \rangle$  and hence  $\Delta\omega = \alpha$ . Therefore, we have that  $(H^p)^\perp = \Delta(E^p(M))$  as claimed, which finishes the proof of the theorem.  $\square$

To complete the proof of the above theorem, we must prove the above two lemmas. We begin with a discussion of the regularity theorem, for which we must now develop some of the theory for Sobolev spaces on compact Riemannian manifolds.

## 4 Sobolev Spaces

In order to begin our discussion of Sobolev spaces, we must introduce some definitions. We shall be using multi-index notation  $\alpha = (\alpha_1, \dots, \alpha_n)$  where each  $\alpha_n \in \mathbb{Z}$ . We define  $|\alpha| := \left(\sum_{j=1}^n |\alpha_j|^2\right)^{1/2}$  and  $[\alpha] = \sum_{j=1}^n \alpha_j$ . Using this notation, we define the  $\alpha$ th derivative operator  $D^\alpha$  for each n-tuple  $\alpha$  of non-negative integers by

$$[D^\alpha u := (-i)^{[\alpha]} \frac{\partial^{[\alpha]} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

where the factor of  $-i$  is introduced for later convenience. Let  $\mathcal{P}$  denote the complex vector space containing all  $C^\infty$  functions on  $\mathbb{R}^n$  with values in  $\mathbb{C}$  which satisfy

$$f(z_1 + 2\pi m_1, \dots, z_n + 2\pi m_n) = f(z_1, \dots, z_n)$$

for all  $m_1, \dots, m_n \in \mathbb{Z}$ . We define the fundamental domain of the space  $\mathcal{P}$  to be the set

$$Q := \{x = (x_1, x_2) \in \mathbb{R}^2 : \max(x_1, x_2) \in (0, 2\pi)\}$$

and note that any function in  $\mathcal{P}$  is determined by its values in  $Q$ . Throughout this section, we will introduce a number of different norms on  $\mathcal{P}$ . By  $\|\cdot\|$ , we shall mean the  $L^2(Q)$  norm and let  $\langle \cdot, \cdot \rangle$  denote the corresponding inner product. By  $\|\cdot\|_\infty$  we denote the  $L^\infty(Q)$  norm.

For any  $\phi \in \mathcal{P}$  and  $\xi \in \mathbb{Z}^n$ , we can define the  $\xi$ th Fourier coefficient  $\phi_\xi$  by

$$\phi_\xi = \langle \phi(x), e^{ix \cdot \xi} \rangle$$

and then define the partial sum  $\sum_{S_j} \phi_\xi e^{ix \cdot \xi}$  where  $S_j = \{\xi = (\xi_1, \dots, \xi_n) : \sup_k |\xi_k| < j\}$ . From basic analysis of Fourier series, along with the Weierstrass approximation theorem, we know that the partial sums converge uniformly on  $\mathbb{R}^2$  to the function  $\phi$ . This follows since  $\phi \in C^\infty$ . In general, this holds if we know that  $\phi \in C^k$  where  $k > \lceil \frac{n}{2} \rceil + 1$ . Therefore, the Fourier series of a  $C^\infty$  periodic function  $\phi$  converges uniformly to  $\phi$ , and thus we set

$$\phi(x) = \sum_{\xi \in \mathbb{Z}^n} \phi_\xi e^{ix \cdot \xi}.$$

Henceforth, the sum over  $\mathbb{Z}^n$  will be implied, so we will write this as only a sum over  $\xi$ . By the uniform convergence, it follows from integration by parts that

$$D^\alpha \phi(x) = \sum_{\xi} \xi^\alpha \phi_\xi e^{ix \cdot \xi}$$

where we define  $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ . In particular, notice how the Fourier “transform” converts differential operators to multiplication operators. Notice that since  $\int_Q e^{ix \cdot (\xi_1 - \xi_2)} dx = 0$  for  $\xi_1 \neq \xi_2$ , we have the well known Parseval identity

$$\int_Q |\phi|^2 = \|(\phi_\xi)_\xi\|_{l^2(\mathbb{Z}^n)}$$

and hence it follows that

$$\|D^\alpha \phi\|^2 = \sum_\xi \xi^{2\alpha} |\phi_\xi|^2$$

Hence, it is clear that given any  $N \in \mathbb{N}$ , there exists a constant  $C > 0$  depending only on  $N$  and  $n$  such that

$$c \sum_\xi (1 + |\xi|^2)^N |\phi_\xi|^2 \leq \sum_{|\alpha|=0}^N \|D^\alpha \phi\|^2 \leq \sum_\xi (1 + |\xi|^2)^N |\phi_\xi|^2.$$

In a moment, we will define a norm on a certain space containing  $\mathcal{P}$  using the right hand side of the above inequality, and hence the inequality will essentially prove the equivalence of this norm to the norm defined by the term in the middle.

Now, define  $\mathcal{S}$  to be all sequences in  $\mathbb{C}$  indexed by n-tuples of integers, i.e. by  $\mathbb{Z}^n$ . In other words, if  $u \in \mathcal{S}$ , then  $u = (u_\xi)_\xi$  where  $\xi$  runs over all of  $\mathbb{Z}^n$  and where each  $u_\xi \in \mathbb{C}$ . We define for each  $s \in \mathbb{Z}$  the Sobolev space  $H_s$  by

$$H_s := \{u \in \mathcal{S} : \sum_\xi (1 + |\xi|^2)^s |u_\xi|^2 < \infty\}.$$

To give motivation for such a space, lets consider an example from the classical scalar case.

**Example 2.** Let  $U \subseteq \mathbb{R}^n$  be a bounded domain. Let  $f \in L^2(U)$  and consider the linear PDE

$$\begin{aligned} -\Delta u &= f \text{ in } U \\ u|_{\partial U} &= 0 \end{aligned}$$

We want to try to prove that there exists a solution to this equation. It is somewhat natural to suspect that such a solution  $u$  should belong to the space  $H = \{v \in C^2(\bar{U}) : v|_{\partial U} = 0\}$ . Note however that  $H$  is not complete, yet we can define the following inner product on  $H$

$$(u, v) := \int_U \nabla u \cdot \nabla v \, dx$$

which naturally induces a norm which we denote by  $\|\cdot\|_H$ . We now claim that  $v \mapsto \int_U f v \, dx$  is a bounded linear functional on  $H$ . Indeed, notice that

$$\begin{aligned} \left| \int_U f v \, dx \right| &\leq \|f\|_{L^2(U)} \|v\|_{L^2(U)} \\ &\leq C \|f\|_2 \|\nabla v\|_2 \\ &= C \|f\|_2 \|v\|_H \end{aligned}$$

where we have used a Poncarè inequality mentioned above. Thus, IF  $H$  were a Hilbert space, the Riesz representation theorem would imply the existence of a unique  $u \in H$  such that for every  $v \in H$

$$\int_U f v \, dx = (u, v) = - \int_U (\Delta u) v \, dx$$

by assuming  $u$  is smooth and using Greens formula (higher dimensional analogue of integration by parts). Hence,  $f = -\Delta u$  in the weak sense. Now, to make this rigorous, we should not work in the space  $H$  but should rather work in the completion of  $H$  under the norm  $\|\cdot\|_H$ . This space is exactly the Sobolev space of all  $H_2$  functions which “vanish” on the boundary of  $U$ .

To further motivate our definition of  $H_s$ , notice that repeated integration by parts implies that if  $\phi \in C^k(\mathbb{R}^n; \mathbb{C})$ , then there exists a constant  $C > 0$  such that

$$|\phi_\xi| \leq \frac{C}{(1 + |\xi|^2)^k}$$

for all  $\xi \in \mathbb{Z}^n$ . Since the series  $\sum_\xi (1 + |\xi|^2)^{-k}$  converges for  $k \geq \lceil \frac{n}{2} \rceil + 1$ , it follows that the more differentiable a function is, the higher the order  $s$  of the Sobolev space that the corresponding sequence  $(\phi_\xi)$  belongs. Indeed, notice that if  $\phi \in C^k$ , then

$$\sum_\xi (1 + |\xi|^2)^s |\phi_\xi|^2 \leq C \sum_\xi (1 + |\xi|^2)^{s-k}$$

which converges if  $k \geq \lceil \frac{n}{2} \rceil + (s + 1)$ , and hence  $\phi \in H_s$  for all  $s$  satisfying this inequality. Hence, the Sobolev spaces are in some sense measuring the regularity of functions, although a general element of  $H_s$  may not even correspond to a function, let alone a continuous function, at all.

From the Schwarz inequality, it follows that the sesquilinear form

$$\langle \phi, \psi \rangle := \sum_\xi (1 + |\xi|^2)^s \phi_\xi \overline{\psi_\xi}$$

defines an inner product on  $H_s$ . Hence, we define the norm  $\|\cdot\|_s$  on  $H_s$  to be the norm induced on  $H_s$  by the above inner product. Clearly,  $H_s$  is just the space  $l^2(\mathbb{Z}^n)$  where the measure is counting measure weighted by  $(1 + |\xi|^2)^s$ . Therefore, we have that  $H_s$  is a Hilbert space under the above inner product.

Notice that by associating each  $\phi \in \mathcal{P}$  with its sequence of Fourier coefficients  $(\phi_\xi)_\xi$ , we can identify  $\mathcal{P}$  with a subspace of  $\mathcal{S}$ . In particular, we can extend the derivative operator to  $\mathcal{S}$  by defining

$$(D^\alpha \phi)_\xi = \xi^\alpha \phi_\xi$$

for each  $\phi \in \mathcal{S}$ . Moreover, notice that  $\mathcal{P} \subset H_s$  for any  $s$ , but there is no reason why you should expect a given element of  $H_s$  to be continuous, or even a function for that matter. Thus, we consider elements of  $H_s$  as formal Fourier series and as “generalized functions”. In the classical case, one must use the theory of distributions and the reader can consult any elementary book on Harmonic analysis for more details. Moreover, if the sequence  $(\phi_\xi)$  is essentially zero, i.e. only finitely many of the terms are non-zero, then this sequence corresponds to an element of  $\mathcal{P}$ . It follows that  $\mathcal{P}$  is dense in  $H_s$  for each  $s$ .

As noted above, the Sobolev spaces  $H_s$  can be thought of as a tool for measuring the regularity of a given “generalized function”. A fundamental lemma due to Sobolev makes this even more clear. Indeed, it says that if  $\phi \in H_s$  for  $s$  sufficiently large, then the formal Fourier series corresponding to  $u$  actually converges to a continuous function possessing a certain number of classical derivatives depending on  $s$ . This lemma is a crucial part in proving the regularity theorem since it allows us to say that a generalized solution of a PDE which belongs to a sufficiently high Sobolev space is actually a classical solution. Before we state the Sobolev lemma, we need the following lemma.

**Lemma 3.**  $D^\alpha$  is a bounded operator from  $H_{s+[\alpha]}$  to  $H_s$  for each  $s$ .

*Proof.* Since we have the inequality  $\xi^{2\alpha} \leq (1 + |\xi|^2)^{[\alpha]}$ , we see that

$$\begin{aligned} \|D^\alpha \phi\|_s^2 &= \sum_{\xi} (1 + |\xi|^2)^s \xi^{2\alpha} |\phi_{\xi}|^2 \\ &\leq \sum_{\xi} (1 + |\xi|^2)^{s+[\alpha]} |\phi_{\xi}|^2. \end{aligned}$$

□

**Lemma 4** (Sobolev Lemma). *If  $s > [\frac{n}{2}] + 1$  and  $\phi \in H_s$ , then the series  $\sum_{\xi} \phi_{\xi} e^{ix \cdot \xi}$  converges uniformly on all of  $\mathbb{R}^n$ . In particular, each  $u \in H_s$  with  $s > [\frac{n}{2}] + 1$  corresponds to a continuous function*

*Proof.* It is enough to prove the series converges absolutely. Well, Cauchy-Schwarz implies

$$\begin{aligned} \sum_{|\xi| < N} |\phi_{\xi}| &\leq \left( \sum_{|\xi| < N} (1 + |\xi|^2)^{-s} \right)^{1/2} \left( \sum_{|\xi| < N} (1 + |\xi|^2)^s |\phi_{\xi}|^2 \right)^{1/2} \\ &\leq \left( \sum_{|\xi| < N} (1 + |\xi|^2)^{-s} \right) \|\phi\|_s. \end{aligned}$$

The result now follows easily from our above discussion and letting  $N \rightarrow \infty$ . □

**Corollary 1.** *If  $u \in H_s$  where  $s > [\frac{n}{2}] + 1 + m$ , then  $D^\alpha \phi = \sum_{\xi} \xi^\alpha \phi_{\xi} e^{ix \cdot \xi}$  converges uniformly for  $[\alpha] \leq m$ . Thus, each  $\phi \in H_s$  for this range of  $s$  corresponds to a function of class  $C^m$ .*

*Proof.* It follows from the fact that  $D^\alpha : H_{s+[\alpha]} \rightarrow H_s$  is a bounded linear map that  $D^\alpha \phi \in H_{s-[\alpha]}$ , where  $s - [\alpha] \geq [\frac{n}{2}] + 1$  by our assumption that  $[\alpha] \leq m$ . Thus, the Sobolev lemma implies that  $\sum_{\xi} \phi_{\xi} e^{ix \cdot \xi}$  converges uniformly. Since this series is the  $\alpha$ th “formal” derivative of the series  $\sum_{\xi} \phi_{\xi} e^{ix \cdot \xi}$ , we see that  $\sum_{\xi} \phi_{\xi} e^{ix \cdot \xi}$  is of class  $C^m$ . □

So, the Sobolev lemma (and its corollary) tell us that if a function belongs to a sufficiently high Sobolev space, then it is indeed continuous. In particular, if a function belongs to  $H_s$  for all  $s > 0$ , then it is indeed smooth.

Our next order of business is to determine when weak solution does indeed lie in a sufficiently high Sobolev space. As motivation, consider the following example.

**Example 3.** Consider our model problem

$$-\Delta u = f \text{ in } \mathbb{R}^n,$$

where  $f \in L^2(\mathbb{R}^n)$ . We assume for formal purposes that  $u$  is smooth and vanishes rapidly enough as  $|x| \rightarrow \infty$  to justify the following calculations. Then

$$\begin{aligned} \int_{\mathbb{R}^n} f^2 dx &= \int_{\mathbb{R}^n} (\Delta u)^2 dx \\ &= \sum_{i,j=1}^n \int_{\mathbb{R}^n} u_{x_i x_i} u_{x_j x_j} dx \\ &= - \sum_{i,j=1}^n \int_{\mathbb{R}^n} u_{x_i x_i x_j} u_{x_j} dx \\ &= \sum_{i,j=1}^n \int_{\mathbb{R}^n} u_{x_i x_j} u_{x_i x_j} dx \\ &= \int_{\mathbb{R}^n} |D^2 u|^2 dx \\ &\geq C \|u\|_{H^2} \end{aligned}$$

for some  $C > 0$ . Thus, knowing that  $f \in L^2$  allowed us to say that, in some sense,  $u$  has two more derivatives than  $f$  does. Moreover, by differentiating the original differential equation (assuming  $f$  has some derivatives), we see that if  $f \in H_s$ , then  $u$  should belong to  $H_{s+2}$ .

In general though, this calculation does not hold since we only begin with a weak solution  $u$ . We thus need to find a way to recapture the above result without explicitly assuming any smoothness properties on  $u$ . To do this, we must first review some fundamental theorems concerning periodic elliptic differential operators.

## 5 Elliptic Differential Operators

In order to understand where the results in this section are heading, it is time we give a glimpse into the strategy for the proof of the regularity theorem. Consider the differential equation

$$\Delta u = \alpha$$

in the space of all smooth  $p$ -forms on a compact oriented Riemannian manifold  $M$ . Let  $U$  be a coordinate patch on  $M$  with coordinate map  $\gamma$  with  $\gamma(U) = \mathbb{R}^n$ . In this coordinate system, differentiable  $p$ -forms become vector-valued functions from  $\mathbb{R}^n$  to  $\mathbb{R}^m \subset \mathbb{C}^m$  with  $m = C(n, p)$ . Hence,  $p$ -forms on  $M$  yield elements of  $C^\infty$ . Moreover, the Laplacian induces a partial differential operator  $L$  of order 2 on  $C^\infty$ . Now, given a point  $p \in \mathbb{R}^n$  we will do all of our analysis in a small neighborhood  $W_p$  of  $p$ . We claim there exists a small neighborhood  $V$  of  $p$  and a periodic partial differential operator  $\tilde{L}$  which agrees with  $L$  on  $V$ . This allows the proof of the regularity theorem to be reduced to a purely local problem. To verify the claim, let  $L_0$  denote the constant coefficient differential operator obtained by evaluating the

coefficients of  $L$  at  $p$ . Since the coefficients of  $L$  are smooth, given any  $\varepsilon > 0$  there exists a neighborhood  $U$  of  $p$  such that the coefficients of  $L$  vary by no more than  $\varepsilon$  on  $U$ . Let  $V \subset \bar{V} \subset U$  and  $\phi$  a smooth function such that  $0 \leq \phi \leq 1$ ,  $\phi = 1$  on  $V$  and  $\phi = 0$  outside a neighborhood of  $U$ . Then the operator  $\tilde{L} = \phi \cdot L + (1 - \phi)L_0$  can clearly be extended to a periodic differential operator on  $C^\infty$ . What will be important is the fact that  $L$  is what is known as a periodic *elliptic* operator. This section is devoted proving that there exists a constant  $C > 0$  such that

$$\|u\|_{s+2} \leq C (\|Lu\|_s + \|u\|_s)$$

whenever  $L$  is an elliptic operator on  $\mathcal{P}$ ,  $u \in H_s$ , and  $Lu \in H_s$ . In other words, if we are studying the equation  $\Delta u = \alpha$  and we know  $\alpha \in H_s$  and  $u$  is a weak solution in  $H_s$ , then in fact  $u \in H_{s+2}$ .

To begin, we first make the following definitions.

**Definition 3.** A (linear) differential operator  $L$  of order 2 on the  $\mathbb{C}$ -valued  $C^\infty$  functions on  $\mathbb{R}^n$  is an operator of the form

$$L = \sum_{[\alpha]=0}^2 a_{i,j}^\alpha D^\alpha$$

where the  $a_{i,j}^\alpha$  are  $C^\infty$  complex valued functions on  $\mathbb{R}^n$ , with at least one  $a_{i,j}^\alpha \neq 0$  for some  $i, j$  for some  $\alpha$  with  $[\alpha] = 2$  (in other words,  $L$  should legitimately be an operator of order 2). A differential operator is said to be a periodic differential operator, or an operator on  $\mathcal{P}$ , if the  $a_{i,j}^\alpha$  are periodic functions.

Notice that by the fact that the map  $D^\alpha : H_{s+[\alpha]} \rightarrow H_s$  is bounded, we expect that if  $L$  is a periodic differential operator of order 2 then

$$\|Lu\|_s \leq C \|u\|_{s+2}$$

for some constant  $C > 0$ .

Now, an important thing to note is that if  $L$  is the periodic differential operator on  $\mathbb{R}^2$  induced by  $\Delta$  (see above), then  $L$  satisfies a VERY important property known as ellipticity

**Definition 4.** A differential operator is called elliptic at  $x \in \mathbb{R}^2$  if

$$\sum_{[\alpha]=2} a_\alpha(x) \xi^\alpha \neq 0$$

for all  $\xi \in \mathbb{R}^2$ . Moreover  $L$  is elliptic on  $\mathbb{R}^2$  if it is elliptic at every point  $x \in \mathbb{R}^2$ .

**Remark 5.** Physically, the ellipticity condition implies that, in a diffusive process, chemicals flow from areas of higher concentration to those of lower concentration.

Moreover, it should be noted that  $\Delta = \partial_x^2 + \partial_y^2$  is elliptic on  $\mathbb{R}^2$ . Indeed, for any  $\xi \in \mathbb{R}^2$ , one computes the above sum to be equal to  $|\xi|^2$ , and hence is non-zero for all  $\xi \neq 0$ .

For such operators, the following lemma gets us one step closer to our desired goal.

**Lemma 5.** Let  $L$  be an elliptic differential operator on  $\mathcal{P}$  of order 2 (in particular,  $L$  is periodic). Then given any  $s \in \mathbb{Z}$ , there exists a constant  $C > 0$  such that

$$\|u\|_{s+2} \leq C (\|Lu\|_s + \|u\|_s)$$

for all  $u \in H_{s+2}$ .

**Remark 6.** First, note that by definition,  $H_s \subset H_t$  whenever  $t < s$ , so if  $u \in H_{s+2}$  then clearly  $u \in H_s$ . Secondly, notice that the above lemma almost gives us the result we want. We would like to say that if we know  $u \in H_s$  and  $Lu \in H_s$ , then  $u \in H_{s+2}$ . However, this result gives us the right machinery needed to prove such a result.

*Proof.* We only prove this in the following special case. Let  $L_0$  be a constant coefficient elliptic operator on  $\mathcal{P}$  of the form

$$L_0 = \sum_{|\alpha|=2} a_\alpha D^\alpha =: P(D)$$

where  $a_\alpha \in \mathbb{C}$ . The fact that  $L_0$  is elliptic implies that for all  $u, \xi \in S^1$ , there exists a constant  $c > 0$  such that  $|P(\xi)u|^2 > c$  (since  $S^1$  is compact). Thus, scaling implies that for all  $u, \xi \in \mathbb{R}^2$ ,

$$|P(\xi)u|^2 > c|\xi|^4|u|^2.$$

Given  $\phi \in \mathcal{P}$ , it follows that

$$\begin{aligned} \|L_0\phi\|_s^2 &= \sum_{\xi} |P(\xi)\phi_\xi|^2 (1 + |\xi|^2)^s \\ &= c \sum_{\xi} |\phi_\xi|^2 |\xi|^4 (1 + |\xi|^2)^s \end{aligned}$$

and hence

$$\begin{aligned} (\|L_0\phi\|_s + \|\phi_0\|_s)^2 &\geq \|L_0\phi\|_s^2 + \|\phi\|_s^2 \\ &= \sum_{\xi} (|P(\xi)\phi_\xi|^2 + |\phi_\xi|^2) (1 + |\xi|^2)^s \\ &\geq \sum_{\xi} (1 + c|\xi|^4) |\phi_\xi|^2 (1 + |\xi|^2)^s \\ &\geq c \sum_{\xi} (1 + |\xi|^2)^{s+2} |\phi_\xi|^2. \end{aligned}$$

□

With this in mind, we are ready to prove the regularity theorem for periodic elliptic differential operators.

**Theorem 4.** Let  $L$  be a periodic elliptic differential operator of order 2,  $u \in \bigcup_{s \in \mathbb{Z}} H_s =: H_{-\infty}$  and  $v \in H_t$ . Then if  $Lu = v$ , it follows that  $u \in H_{t+2}$ .

*Proof.* Since  $\mathcal{P}$  is dense in  $H_s$  for all  $s$ , we will work only with  $u \in \mathcal{P}$ . In particular we will show that if  $u \in H_s$ ,  $v \in H_{s-2+1} = H_{s-1}$ , then  $u \in H_{s+1}$ . In order to do this, we work with difference quotients (which is a very valuable tool in PDE's by the way). For each  $h \in \mathbb{R}^2 - \{0\}$ , let

$$u^h(x) = \frac{u(x+h) - u(x)}{|h|}$$

and let  $L^h$  denote the differential operator resulting from replacing all coefficients of  $L$  with their difference quotients. Then an easy calculation shows you that

$$L(u^h) = L(u)^h - L^h(T_h u)$$

where  $T_h u(x) = u(x + h)$ . Therefore, by the above lemma we have that

$$\begin{aligned} \|u^h\|_s &\leq C (\|L(u^h)\|_{s-2} + \|u^h\|_{s-2}) \\ &\leq C (\|(Lu)^h\|_{s-2} + \|L^h(T_h u)\|_{s-2} + \|u^h\|_{s-2}). \end{aligned}$$

Since the coefficients of  $L$  are smooth, their difference quotients are uniformly bounded and hence there is a  $C > 0$  independent of  $h$  such that

$$\|L^h(T_h u)\|_{s-2} \leq C \|T_h u\|_s = C \|u\|_s$$

since  $L$  is of order 2. Using the fact that  $\|u^h\|_s \leq \|u\|_{s+1}$  (which can be easily verified, it follows that  $\|(Lu)^h\|_{s-2} \leq \|Lu\|_s$  and hence

$$\|u^h\|_s \leq C (\|Lu\|_{s-1} + \|u\|_s).$$

Since the right hand side is independent of  $h$ , taking  $h \rightarrow 0$  yields

$$\|u\|_{s+1} \leq C (\|Lu\|_{s-1} + \|u\|_s).$$

□