Introduction to the Inverse Scattering Transform

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Study of solitary waves began with observations by J. Scott Russell:
"a large solitary elevation, a rounded, smooth, and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.... Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel." (Russell 1838)

Russell found that no mathematical theory available at the time predicted a solitary wave, but noted that
"it now remained to the mathematician to predict the discovery after it had happened, i.e. to give an a priori demonstration a posteriori." (Russell 1844)
Inverse scattering theory gives such a prediction (1967), (1974).

(Motivation)Consider the IVP

$$\begin{aligned} u_t &= k u_{xx} \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x) \end{aligned}$$

Separate variables: u(x,t)=X(x)T(t)

$$\begin{aligned} \Rightarrow \quad X'' &+ \lambda X = 0 \\ T' &+ \lambda T = 0 \end{aligned}$$

Find eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

$$u(x,t) = \sum_{n \in \mathbb{Z}} \underbrace{\left(\underbrace{\widehat{f}(n)}_{F.T. of Initial Data} \qquad e^{k\lambda_n t} \right)}_{Time Evolution of F.T.} e^{i\sqrt{\lambda_n} x}$$

Notice $\frac{d\lambda_n}{dt} = 0$, i.e. the eigenvalues are constants of motion.

⇒ Means you don't have to resolve the spectral problem for each time t!

When solving non-linear PDE, would like to convert to a spectral problem

$$L(t)v = \lambda v$$

such that $\sigma(L(t)) = \sigma(L(0))$, as well as a time evolution problem

$$v_t = Mv.$$

Consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \;\; (x,t) \in \mathbb{R} \times \mathbb{R}^+$$

Used to model shallow water waves (such as that observed by Russell).

Theorem 1. *If a one-parameter family of potentials* $q(x)(t) := u(x,t)$ *evolve according to the KdV equation, then the coresponding family of Schödinger operators*

$$L(t) = -\partial_x^2 + q(x)(t) \;\; \textit{on } L^2(\mathbb{R})$$

satisfy $\sigma(L(t)) = \sigma(L(0))$, *given an appropriate time evolution equation.*

To see how this helps one solve the KdV, we first solve the spectral problem for $L := L(0)$.

Suppose $q(x) := q(x, 0) \in C_0^\infty(\mathbb{R})$, so that $q(x) \to 0$ as $|x| \to \infty$. Consider

$$-u'' + q(x)u = \lambda u \tag{1}$$

on $L^2(\mathbb{R})$. As $|x| \to \infty$, (1) looks like

$$-u'' = k^2\, u \quad (\lambda = k^2)$$

Define special solutions ("Jost Solutions" or "Scattering States") of (1) by

$$\begin{cases} \phi(x) \sim e^{-ikx}, & x \to -\infty; \\ \psi(x) \sim e^{ikx}, & x \to \infty. \end{cases}$$

$\lambda < 0 \Rightarrow k \in \mathbb{R}i \Rightarrow \phi, \psi$ decay exponentially as $x \to \pm\infty$ for $Im(k) > 0$.

Question: Why do ψ and ϕ exist?

1. Clear if q has compact support.
2. In general, let $v(x) := e^{-ikx}\psi(x)$

   $$\Rightarrow v(x) = 1 - \frac{1}{2ik}\int_x^\infty \left(1 - e^{2ik(x-y)}\right) q(y)v(y)dy$$

Define Neumann iterates

$$\begin{cases} v_1(x) = 1, \\ v_{n+1}(x) = 1 - \underbrace{\frac{1}{2ik}\int_x^\infty \left(1 - e^{2ik(x-y)}\right) q(y)v_n(y)dy}_{T(v_n)} \end{cases}$$

T is a contraction! So $\{v_n\}$ is Cauchy in L^∞, and hence converges to a bounded function $v(x)$ which satisfies the above integral equation and $u(x) \to 1$ as $x \to \infty$.

$\Rightarrow \psi(x) = e^{ikx}u(x)$ satisfies Schrödingers equation (1) and $\psi \sim e^{ikx}$ as $x \to \infty$.

Now, define $\widetilde{\psi}(x;k) := \psi(x;-k)$

$$\widetilde{\psi}(x;k) \sim e^{-ikx}, \text{ as } x \to \infty.$$

$$W(\psi,\widetilde{\psi})(x) = \lim_{x\to\infty} W(\psi,\widetilde{\psi})(x) = 2ik$$

$\Rightarrow \psi$, $\widetilde{\psi}$ form a solution basis for (1)

There exist functions a, $b : \mathbb{R} \to \mathbb{C}$ such that

$$\underbrace{\phi(x)}_{\sim e^{-ikx},\, x\to-\infty} = a(k) \underbrace{\widetilde{\psi}(x)}_{\sim e^{-ikx},\, x\to\infty} + b(k) \underbrace{\psi(x)}_{\sim e^{ikx},\, x\to\infty}$$

$$\begin{aligned} a(k) = 0 \;&\Rightarrow\; \phi \sim e^{-ik|x|},\ x \to \infty \\ &\Rightarrow\; \phi \in L^2(\mathbb{R}) \\ &\Rightarrow\; \lambda = k^2 \text{ is an e.v.} \end{aligned}$$

Converse holds as well.

Facts:

(1) $a(k)$ is analytic in UHP

(2) $e^{ikx}\phi$ and $e^{-ikx}\psi$ are analytic in UHP

(3) $\exists$ finitely many $k_j \in \mathbb{R}^+ i$ such that k_j^2 is an e.v.

Finding $a(k)$, $b(k)$ and $\{k_j\}$ above gives direct scattering transform of KdV at $t = 0$.

Example 1 (Scattering by Delta-potential). *Consider*

$$u'' + (k^2 - \beta\delta(x))u = 0,\ x \in \mathbb{R}$$

$$\Rightarrow u(x;k) = \begin{cases} e^{-ikx}, & \textit{for } x < 0; \\ a(k)e^{-ikx} + b(k)e^{ikx}, & \textit{for } x > 0. \end{cases}$$

To calculate $a(k)$ *and* $b(k)$*, require*
(1) u *to be continuous at* $x = 0$

$$\Rightarrow a + b = 1$$

(2) Jump condition $u'|_{-\varepsilon}^{\varepsilon} - \beta u(0) = 0$

$$\Rightarrow b(k) = \frac{\beta}{2ik},\ \ a(k) = \frac{2ik - \beta}{2ik}$$

$a(k) = 0$ *when* $k = -i\beta/2$.
$\beta < 0 \Rightarrow u(x) = e^{-|\beta x|/2}$ *is eigenfunction (bound-state)*
$\beta > 0 \Rightarrow$ *No bound states!*

To continue solving KdV, solve corresponding time evolution equation (TRIVIAL!!!)

$$a(k,t) = a(k,0), \quad b(k,t) = e^{8ik^3t}b(k,0)$$

Q:How do we recover $q(x)(t) = u(x,t)$ from the above scattering data?

Fix $t > 0$ and recall $\psi(x;k) \sim e^{ikx}$ as $x \to \infty$. Assume

$$\psi(x;k) = \underbrace{e^{ikx}}_{FreeEvolution} + \underbrace{\int_x^\infty A(x,y)e^{iky}dy}_{BackScatteringFromPotential}$$

$$\begin{aligned}\int_x^\infty \left(A_{xx}(x,y) - A_{yy}(x,y) - q(x)A(x,y)\right)e^{iky}dy \\ + \lim_{y\to\infty}\left(A_y(x,y) - ikA(x,y)\right) \\ - \left(2\frac{d}{dx}A(x,x) + q(x)(t)\right)e^{ikx} = 0\end{aligned}$$

Require
(1) $A_{xx}(x,y) - A_{yy}(x,y) - q(x)A(x,y) = 0$
(2) $\lim_{y\to\infty}\left(A_y(x,y) - ikA(x,y)\right) = 0$
(3) $\frac{d}{dx}A(x,x) = \frac{1}{2}q(x)(t)$

Can prove such an A exists an is unique $\Rightarrow$ Find A, find $q(x)(t) = u(x,t)$!!!

Let C be a contour in $\mathbb{C}$ starting at $-\infty + 0i$, ending at $\infty + 0i$ and passing above all the zeroes of $a(k)$. Define

$$\begin{aligned} r(x) &= \frac{1}{2\pi}\int_C \frac{b(k)}{a(k)}e^{ikx}dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\frac{b(k)}{a(k)}}_{\rho(k)} e^{ikx}dk + \sum_j \underbrace{\frac{b(k_j)}{a'(k_j)}}_{C_j} e^{ik_jx} \end{aligned}$$

GLM Equation:

$$0 = r(x+y) + \int_x^{\infty} r(s+y)A(x,s)dx + A(x,y)$$

for $y > x$. Solve using Fredholm alternative.

Can solve GLM equation explicitly in special cases where $b(k) = 0$ for $k \in \mathbb{R}$ (vanishing reflection coefficient!) Potentials which correspond to vanishing reflection coefficient usually give rise to **N-Soliton solutions!**

Example 2. *The KdV equation*

$$\begin{cases} u_t + 6uu_x + u_{xxx} = 0, \ (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = 2\,\mathrm{sech}^2\, x \end{cases}$$

gives rise to one bound state of the operator $L(0) = -\partial_x^2 + 2\,\mathrm{sech}^2\, x$, *namely* $\lambda = -1$. *In this case* $b(k) = 0$ *on* $\mathbb{R}$ *and the GLM equation can be solved in closed form, yielding*

$$u(x,t) = 2\,\mathrm{sech}^2(x - 4t)$$

which is a 1-soliton solution of KdV. This solution has all properties initially observed by Russell in 1838.

Question: Why does this procedure work at all?

Answer: KdV has Hamiltonian formulation

$$\begin{aligned} \frac{dp}{dt} &= \frac{d}{dx}\frac{\delta H}{\delta p} \\ H &= -\int \left(p^3 - \frac{1}{2} p_x^2 \right) dx \end{aligned}$$

Poisson Brackets:

$$\langle A, B \rangle := \int_{-\infty}^{\infty} \frac{\delta A}{\delta p} \frac{\partial}{\partial x} \frac{\delta B}{\delta p} dx$$

Transformation $(p, q) \to (P, Q)$ is canonical if

$$\begin{aligned} \langle Q(x), Q(y) \rangle &= 0, \ \ \langle P(x), P(y) \rangle = 0 \\ \langle P(x), Q(y) \rangle &= \delta(x - y). \end{aligned}$$

Lets define a subset S of the scattering data, from which the remaining scattering data can be reconstructed. For $k \in \mathbb{R}$, define

$$\begin{aligned} P(k) : &= \frac{k}{\pi} \log |a(k)|^2, \ \ P_n := -2k_n^2 \\ Q(k) : &= arg \left(\frac{b(k)}{a(k)} \right) = arg(\rho(k)), \ \ Q_n := log(C_n) \end{aligned}$$

The mapping

$$q(x) := u(x,0) \to \underbrace{\{P(k), Q(k), P_n, Q_n\}}_{S}$$

is canonical, and turns out to be of action-angle type. Hamilton's equations in these variables are equivalent to

$$\begin{aligned} a(k,t) &= a(k,0) \\ b(k,t) &= b(k,0)e^{8ik^3t} \\ C_n(t) &= C_n(0)e^{8k_n^3t} \end{aligned}$$

Main Point: We may view the IST as a concrete method to effect a canonical transformation into action-angle variables. The dynamics of the system are then very simple when described in terms of these variables.