Spectral Theorem for Normal Unbounded Operators*

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The purpose of this write-up is to extend the Spectral Theorem for bounded normal operators to cover unbounded operators. The proof relies heavily on the Spectral Theorem for bounded normal operators, which we should all be familiar with by now. The proof of the Spectral Theorem involves several preliminary lemmas whose proofs should probably be omitted upon first reading. We then progress to give an application of the spectral theorem to strongly continuous unitary groups of operators. In particular, we prove Stone's Theorem, which answers the question, "When is a strongly continuous group of unitary operators on a Hilbert space generated by a self-adjoint operator?" This question has many applications in mathematics and physics where we may consider the solution operator for the Schrödinger equation as such a unitary group.

To begin, let us recall the following definition.

Definition 1. *A linear operator N on a Hilbert space H is normal if N is closed, densely defined, and $NN^* = N^*N$.*

In particular, if N is a normal operator, then $Dom(N^*N) = Dom(NN^*)$. As in the bounded operator case, it is clear that any self-adjoint operator is normal. The initial goal of this write-up is to present a proof of the following theorem.

Theorem 1 (The Spectral Theorem)**.** *If N is a normal operator on H, then there exists a unique spectral measure E defined on $B_{\mathbb{C}}$ such that*

1. $N = \int z \, dE(z)$;
2. $E(\Delta) = 0$ *if* $\Delta \bigcap \sigma(N) = \emptyset$;
3. *if* $U \subseteq \mathbb{C}$ *is open and* $U \bigcap \sigma(N) \neq \emptyset$, *then* $E(U) \neq 0$;
4. *if* $A \in B(H)$ *is such that* $AN \subseteq NA$ *and* $AN^* \subseteq N^*A$, *then* $A\left(\int \phi dE\right) \subseteq \left(\int \phi dE\right) A$ *for every Borel function* ϕ *on* $\mathbb{C}$.

First, we make a few comments as to the strategy of the proof. Suppose we have a spectral measure E defined on $B_{\mathbb{C}}$, and let $N = \int z \, dE(z)$. For $0 \leq a \leq b < \infty$, define $\Delta = \{z \in \mathbb{C} : |z| \in [a, b]\}$ and

$$\begin{aligned} H_\Delta &= E(\Delta)H \\ &= \{h \in Dom(N) : h \in Dom(N^n) \ \forall n \in \mathbb{N} \text{ and } a^n||h|| \leq ||N^n h|| \leq b^n||h||\}. \end{aligned}$$

*Adapted from Conway's book "A Course in Functional Analysis", 2^{nd} Edition.

We will show that H_Δ is a closed subspace of H which reduces N and for which $N|_{H_\Delta}$ is a bounded normal operator. Once we have this, we decompose $\mathbb{C}$ into the disjoint union of annuli $\{\Delta_j\}$ and prove that each $H_j = E(\Delta_j)H$ is a closed reducing subspace for N, $N_j := N|_{H_j}$ is a bounded normal operator, and that $H = \bigoplus_j H_j$. We then apply the Spectral Theorem for bounded normal operators to each of the N_j, and then (by the following lemma) we can use direct sums to obtain a spectral measure for the normal operator N.

1 Preliminary Results

We now prove the technical lemmas which will allow us to make the above remarks into an actual proof. The following lemma tells us, once we have applied the (bounded) Spectral Theorem to each N_j, how to "glue" the results together to give us information about our original operator N.

Lemma 1. *Let H_1, H_2, ... be Hilbert spaces and let $A_n \in B(H_n)$ for all $n \geq 1$. If $D := \{(h_n) \in \bigoplus_n H_n : \sum_{n=1}^\infty ||A_n h_n||^2 < \infty\}$ and A is defined on $H := \bigoplus_n H_n$ by $A(h_n) = (A_n h_n)$ whenever $(h_n) \in D$, then A is a closed operator on H. Moreover, A is a normal operator if and only if each A_n is normal.*

Proof. Since $H_n \subseteq D$ for each n and $H = \bigoplus_n H_n$, it is clear that D is dense in H. Since each A_n is linear, it follows that A is a densely defined linear operator. To show that A is closed, suppose we have a sequence $(h^{(j)}) \subseteq Dom(A)$ such that $h^{(j)} \oplus Ah^{(j)} \to h \oplus g$ in $H \bigoplus H$ for some $h, g \in H$. Then for each n, there exist $h_n, g_n \in H_n$ such that $h_n^{(j)} \oplus A_n h_n^{(j)} \to h_n \oplus g_n$. Set $h = (h_n)$ and $g = (g_n)$. Since each A_n is bounded, $A_n h_n = g_n$. Moreover $\sum_n ||Ah_n||^2 = \sum_n ||g_n||^2 = ||g||^2 < \infty$, so $h \in Dom(A)$ and hence $Ah = g$. Thus, A is closed.

Now, by definition we have

$$\begin{aligned} Dom(A^*) &= \{(h_n) \in H : Dom(A) \ni (\phi_n) \to (\langle h_n, A_n \phi_n \rangle) \text{ is a bounded linear functional}\} \\ &= \{(h_n) \in H : \sum_n ||A_n^* h_n||^2 < \infty\}. \end{aligned}$$

By making the observation that $A^*(h_n) = (A_n^* h_n)$ when $(h_n) \in Dom(A^*)$, we have that A is normal if and only if

$$(A_n A_n^*) = AA^* = A^*A = (A_n^* A_n),$$

i.e. A is normal if and only if each A_n is normal. $\square$

Lemma 2. *If $N : H \to H$ is densely defined (not necessarily normal) and $J : H \oplus H \to H \oplus H$ is defined by $J(h \oplus k) = (-k) \oplus h$, then J is an isomorphism and*

$$Gra(N^*) = [J\ Gra(N)]^\perp.$$

Proof. Clearly, J is an isomorphism. Now, notice that if $k \in Dom(N^*)$ and $h \in Dom(N)$, then

$$\begin{aligned} \langle k \oplus N^*k, J(h \oplus Nh) \rangle &= \langle k \oplus N^*k, -Nh \oplus h \rangle \\ &= -\langle k, Nh \rangle + \langle N^*k, h \rangle = 0. \end{aligned}$$

Thus, $Gra(N^*) \subseteq [J\ Gra(N)]^\perp$. Conversely, if $k \oplus f \in [J\ Gra(N)]^\perp$, then for every $h \in Dom(N)$, $0 = \langle k \oplus f, -Nh \oplus h \rangle = -\langle k, Nh \rangle + \langle f, h \rangle$. Hence $\langle k, Nh \rangle = \langle f, h \rangle$. Therefore, $k \in Dom(N^*)$ and $N^*k = f$, which completes the proof. $\square$

Proposition 1. *Let N be a closed linear operator. Then*

1. $1 + N^*N$ *has a bounded inverse defined on all of* H*;*
2. *if* $B = (1 + N^*N)^{-1}$*, then* $B \geq 0$ *and* $||B|| \leq 1$*;*
3. *the operator* $C := NB$ *is a contraction, i.e.* $||C|| \leq 1$*;*
4. $BC = CB$*;*
5. N^*N *is self-adjoint and* $\{h \bigoplus Nh : h \in Dom(N^*N)\}$ *is dense in the graph of* N*.*

Proof. As in the above lemma, define $J : H \oplus H \to H \oplus H$ by $J(h \oplus k) = (-k) \oplus h$. Then $Gra(N^*) = [J\ Gra(N)]^\perp$. Hence, given $h \in H$ there exists $f \in Dom(N)$ and $g \in Dom(N^*)$ such that $0 \oplus h = J(f \oplus Nf) + g \oplus N^*g = (-Nf) \oplus f + g \oplus N^*g$. Thus, $0 = -Nf + g$. Moreover, $h = f + N^*g = f + N^*Nf = (1 + N^*N)f$ and therefore $Ran(1 + N^*N) = H$.

Also, for $f \in Dom(N^*N)$, $Nf \in Dom(N^*)$ and $\|(1 + N^*N)f\|^2 = \|f\|^2 + 2\|Nf\|^2 + \|N^*Nf\|^2 \geq \|f\|^2$ and hence $ker(1 + N^*N) = \{0\}$. Therefore, $B = (1 + N^*N)^{-1}$ exists and is defined on all of H. Taking $h = (1 + N^*N)f$ above proves that $\|h\| \geq \|Bf\|$ and hence B is a contraction. In addition,

$$\langle Bh, h\rangle = \langle f, (1 + N^*N)f\rangle = \|f\|^2 + \|Nf\|^2 \geq 0,$$

which proves (1) and (2). By the same argument, if we define $C = NB$ and let $f \in Dom(N^*N)$ and $h = (1 + N^*N)f$ then $\|Ch\|^2 = \|Nf\|^2 \leq \|(1 + N^*N)f\|^2 = \|h\|^2$ and hence C is a contraction. Thus (3) is proved.

For the proof of (4), let $f \in Dom(N^*N)$ be such that $h = (1 + N^*N)f$. Then

$$BCh = BNBh = BNf.$$

The proof will be complete if we can show that $BNf = NBf$ since then $BCh = NBf = NBBh = CBh$. Note that it suffices to prove $BN \subseteq NB$. If $f \in Dom(BN)$, then $f \in Dom(N)$ by definition. Let $g \in Dom(N^*N)$ be such that $f = (1 + N^*N)g$ so that $N^*Ng \in Dom(N)$. Hence, $Ng \in Dom(NN^*) = Dom(N^*N)$ and so

$$Nf = Ng + NN^*Ng = (1 + N^*N)Ng.$$

Therefore $BNf = B(1 + N^*N)Ng = Ng$. Since $NBf = Ng$, it follows that $BN = NB$ on $Dom(N)$, and thus $BC = CB$ as claimed.

To prove (5), notice that since N is closed by hypothesis, it suffices to show that $Gra(N) \cap \{h \oplus Nh : h \in Dom(N^*N)\}^\perp = \{0\}$. Let $g \in Dom(N)$ and suppose that for every $h \in Dom(N^*N)$ we have $0 = \langle g \oplus Ng, h \oplus Nh\rangle$. Then

$$\begin{aligned} 0 &= \langle g, h\rangle + \langle Ng, Nh\rangle \\ &= \langle g, h\rangle + \langle g, N^*Nh\rangle \\ &= \langle g, (1 + N^*N)h\rangle \end{aligned}$$

and hence $g \perp Ran\,(1 + N^*N) = H$, i.e. $g = 0$.

Finally, note that $Dom(N^*N)$ is dense by our above work. Let $f, g \in Dom(N^*N)$ so that $f, g \in Dom(N)$ and $Nf, Ng \in Dom(N^*)$. It follows that

$$\langle N^*Nf, g\rangle = \langle f, N^*Ng\rangle$$

so that N^*N is symmetric. Also, $1+N^*N$ has a bounded inverse and hence $1+N^*N$ is closed, i.e. N^*N is closed. Moreover, $-1 \notin \sigma(N^*N)$ so that $\mathbb{R} \nsubseteq \sigma(N^*N)$ and hence N^*N is self-adjoint as claimed. $\square$

Lemma 3. *With the same notation as above, let $B = \int t dP(t)$ be the spectral representation of B. Let $\delta \in (0,1)$ and let Δ be a Borel subset of $[\delta, 1]$. Then $H_\Delta := P(\Delta)H \subseteq Dom(N)$, H_Δ is invariant for both N and N^*, and $N|_{H_\Delta}$ is a bounded normal operator with*

$$||N|_{H_\Delta}|| \leq \left(\frac{(1-\delta)}{\delta}\right)^{1/2}.$$

Proof. Notice that if $h \in H_\Delta$, then $P(\Delta)h = h$ and hence

$$\begin{aligned} ||Bh||^2 &= \langle Bh, Bh\rangle \\ &= \left\langle B^2h, h\right\rangle \\ &= \left\langle B^2P(\Delta)h, h\right\rangle \\ &= \int_\Delta t^2 dP_{h,h}(t) \\ &\geq \delta^2||h||^2. \end{aligned}$$

So, $ker(B|_{H_\Delta}) = \{0\}$ and hence $B|_{H_\Delta}$ is invertible. Thus, there exists $g \in H_\Delta$ such that $h = Bg$ and hence $H_\Delta \subseteq Ran(B) = Dom(1+N^*N) \subseteq Dom(N)$.

Now, we show that H_Δ is invariant for both N and N^*. Let $h \in H_\Delta$ and let $g \in H_\Delta$ be such that $h = Bg$. Then $Nh = N(Bg) = (NB)g = Cg$ by definition. Since $BC = CB$, we know that $P(\Delta)C = CP(\Delta)$ and therefore $Cg \in H_\Delta$ since $g \in H_\Delta$. It follows that $Nh = Cg \in H_\Delta$ so that H_Δ is an invariant subspace for N. Now, let $M = N^*$ and $B_1 := (1+M^*M)^{-1}$. Notice that since N is normal, $B_1 = B$ and hence the above argument proves that $N^*H_\Delta \subseteq H_\Delta$. Moreover, since N is normal, clearly $N|_{H_\Delta}$ is normal.

Finally, we use the spectral representation of B to prove that $N|_{H_\Delta}$ is bounded. Let $h \in H_\Delta$. Then

$$\begin{aligned} ||Nh||^2 &= \langle N^*Nh, h\rangle \\ &= \langle((1+N^*N)-1)h, h\rangle \\ &= \left\langle(B^{-1}-1)h, h\right\rangle \\ &= \int_\delta^1 \left(t^{-1}-1\right) dP_{h,h}(t) \leq ||h||^2\frac{(1-\delta)}{\delta}. \end{aligned}$$

$\square$

Now, we recall some definitions and make some more of our own. Let (X, Ω) be a measurable space and H a Hilbert space. If E is a spectral measure for (X, Ω, H), then recall that given $h, k \in H$, $E_{h,k}$ is the complex valued measure defined by $E_{h,k}(\Delta) = \langle E(\Delta)h, k\rangle$ for each $\Delta \in \Omega$.

Suppose $\phi : X \to \mathbb{C}$ is an Ω-measurable function and set $\Delta_n := \{x \in X : |\phi(x)| \in [n-1, n)\}$. Then $\chi_{\Delta_n}\phi$ is clearly a bounded Ω-measurable function. Define $H_n := E(\Delta_n)H$ and notice then that $H = \bigoplus H_n$ since $\cup_n \Delta_n = X$ and the $\{\Delta_n\}$ are pairwise disjoint. If we define $E_n(\Delta) = E(\Delta \cap \Delta_n)$, then E_n is a spectral measure for (X, Ω, H). Moreover, the operator $\int \phi \, dE_n$ is normal on H_n. We now want to put these normal operators together using Lemma 1 as a guide. To begin, define

$$D_\phi := \left\{ h \in H : \sum_{n=1}^{\infty} \left\| \left(\int \phi \, dE_n \right) E(\Delta_n) h \right\|^2 < \infty \right\}$$

and define $N_\phi : H \to H$ by

$$N_\phi h = \sum_{n=1}^{\infty} \left(\int \phi \, dE_n \right) E(\Delta_n) h.$$

We will also denote this by $N_\phi = \int \phi \, dE$. By Lemma 1, the operator N_ϕ is normal on D_ϕ. This example essentially contains all of the main ideas of the proof of the Spectral Theorem. Indeed, given a normal operator our strategy is to break $\mathbb{C}$ up into disjoint annuli on which the restriction of our operator become bounded and normal. Then, we construct a spectral measure E by summing (as above) and then consider the function N_ϕ on D_ϕ where $\phi(z) = z$, which will be normal as above. Then, we will use the following theorem in order to prove that N_ϕ agrees with our original normal operator.

Theorem 2. *If E is a spectral measure for (X, Ω, H), $\phi : X \to \mathbb{C}$ is an Ω-measurable function, and D_ϕ and N_ϕ are defined as above, then*

1. $D_\phi = \{h \in H : \int |\phi|^2 \, dE_{h,h} < \infty\}$;
2. *for h in D_ϕ and $f \in H$, $\phi \in L^1(|E_{h,f}|)$ with*

$$\begin{aligned} \int |\phi| d|E_{h,f}| &\leq ||f|| \left(\int |\phi|^2 \, dE_{h,h} \right)^{1/2}, \\ \left\langle \left(\int \phi \, dE \right) h, f \right\rangle &= \int \phi \, dE_{h,f}, \\ \textit{and } \left\| \left(\int \phi \, dE \right) h \right\|^2 &= \int |\phi|^2 \, dE_{h,h}. \end{aligned}$$

Proof. Using the *-homomorphic properties associated with the spectral measure E, we have

$$\begin{aligned} \left\| \left(\int \phi \, dE_n \right) E(\Delta_n) h \right\|^2 &= \left\langle \left(\int \chi_{\Delta_n} \phi \, dE \right)^* \left(\int \chi_{\Delta_n} \phi \, dE \right) h, h \right\rangle \\ &= \int_{\Delta_n} |\phi|^2 dE_{h,h} \end{aligned}$$

by definition. Hence (1) is proved.

Now, let $h \in D_\phi$, $f \in H$. By the Radon-Nikodym Theorem, there is an Ω-measurable function u such that $|u| = 1$ and $|E_{h,f}| = u E_{h,f}$, where $|E_{h,f}|$ denotes the variation for $E_{h,f}$. Let $\phi_n = \sum_{k=1}^{n} \chi_{\Delta_k} \phi$ so that ϕ_n and $u\phi_n$ are bounded. Thus

$$\begin{aligned} \int |\phi_n| d|E_{h,f}| &= \int |\phi_n| u \, dE_{h,f} \\ &= \left\langle \left(\int |\phi_n| u \, dE \right) h, f \right\rangle \\ &\leq \|f\| \left\| \left(\int |\phi_n| u \, dE \right) h \right\|. \end{aligned}$$

But,

$$\begin{aligned}\left\|\left(\int |\phi_n| u\, dE\right)h\right\|^2 &= \left\langle \left(\int |\phi_n| u\, dE\right)h, \left(\int |\phi_n| u\, dE\right)h\right\rangle \\ &= \left\langle \left(\int |\phi_n|^2 dE\right)h, h\right\rangle \\ &= \int |\phi_n|^2 dE_{h,h} \\ &\leq \int |\phi|^2 dE_{h,h}.\end{aligned}$$

Therefore, $\int |\phi_n|\ d|E_{h,f}| \leq \|f\| \left(\int |\phi|^2 dE_{h,h}\right)^{1/2}$ for all n. Letting $n \to \infty$ yields the first equality in part (2).

Since ϕ_n is bounded, we clearly have $\left\langle \left(\int \phi_n\ dE\right)h, f\right\rangle = \int \phi_n\ dE_{h,f}$. If $h \in D_\phi$ and $f \in H$, then by the Lebesgue Dominated Convergence Theorem we have $\int \phi_n\ dE_{h,f} \to \int \phi\ dE_{h,f}$ as $n \to \infty$. But,

$$\begin{aligned}\left(\int \phi_n\ dE\right)h &= \left(\int \phi\ dE\right) E\left(\bigcup_{j=1}^{n} \Delta_j\right) h \\ &= E\left(\bigcup_{j=1}^{n} \Delta_j\right)\left(\int \phi\ dE\right)h.\end{aligned}$$

Since the projections $E\left(\cup_{j=1}^{n} \Delta_j\right)$ converge to $E(X)$ in the strong operator topology as $n \to \infty$, we have

$$\left\langle \left(\int \phi_n\ dE\right)h, f\right\rangle \to \left\langle \left(\int \phi\ dE\right)h, f\right\rangle$$

as $n \to \infty$, which finishes the proof of the theorem. □

We are now prepared to prove the Spectral Theorem.

Proof of the Spectral Theorem. Let B and C be defined as above and let $B = \int t dP(t)$ be the spectral representation of the operator B. Let $P_n := P\left(\left(\frac{1}{n+1}, \frac{1}{n}\right]\right)$ for each $n \geq 1$. Since $ker(B) = \{0\} = P(\{0\})H$, we see that $\sum_n P_n = 1$. Define $H_n := P_n H$. Now, Lemma 3 applies directly and tells us that $H_n \subseteq Dom(N)$, H_n reduces N, and $N_n := N|_{H_n}$ is a bounded normal operator with $||N_n|| \leq n^{1/2}$. Moreover, we clearly have that for all $h \in H_n$,

$$(1 + N_n^* N_n)Bh = h = B(1 + N_n^* N_n)h$$

so that $B|_{H_n} = (1 + N_n^* N_n)^{-1}$. Therefore, if $\lambda \in \sigma(N_n)$, $(1 + |\lambda|^2)^{-1} \in \sigma(B|_{H_n}) \subseteq \left[\frac{1}{n+1}, \frac{1}{n}\right]$ by above. Therefore

$$\sigma(N_n) \subseteq \{z \in \mathbb{C} : (n-1)^{1/2} \leq |z| \leq n^{1/2}\} =: \Delta_n.$$

Now, let $N_n = \int t\ dE_n(t)$ be the spectral representation of N_n and define

$$E(\Delta) = \sum_n E_n\left(\Delta \cap \Delta_n\right)$$

for each Borel subset Δ of $\mathbb{C}$. Notice that $E_n(\Delta \cap \Delta_n)$ is a projection with range in H_n. Moreover, since $H_n \perp H_m$ for $n \neq m$, it is clear that $E(\Delta)$ defines a projection in $B(H)$. The projection E is our candidate for the spectral measure of N.

Claim: E is a spectral measure.

Since each E_n is a spectral measure and since $\cup_n \Delta_n = \mathbb{C}$, it is clear that $E(\emptyset) = 0$ and $E(\mathbb{C}) = 1$. Again, since $H_n \perp H_m$ for $n \neq m$, if $\Lambda_1, \Lambda_2 \in B_{\mathbb{C}}$, then

$$\begin{aligned} E(\Lambda_1 \cap \Lambda_2) &= \sum_n E_n(\Lambda_1 \cap \Lambda_2 \cap \Delta_n) \\ &= \sum_n E_n(\Lambda_1 \cap \Delta_n) E_n(\Lambda_2 \cap \Delta_n) \\ &= \left(\sum_n E_n(\Lambda_1 \cap \Delta_n)\right)\left(\sum_n E_n(\Lambda_2 \cap \Delta_n)\right) \\ &= E(\Lambda_1)E(\Lambda_2). \end{aligned}$$

Now, notice that if $h \in H$, then $\langle E(\Delta)h, h\rangle = \sum_n \langle E_n(\Delta \cap \Delta_n)h, h\rangle$. Thus, if $\{\Lambda_j\}$ is a disjoint family of Borel subsets of $\mathbb{C}$, then

$$\begin{aligned} \langle E(\cup_j \Lambda_j) h, h\rangle &= \sum_n \langle E_n((\cup_j \Lambda_j) \cap \Delta_n) h, h\rangle \\ &= \sum_n \sum_j \langle E_n(\Lambda_j \cap \Delta_n)h, h\rangle \\ &= \sum_j \sum_n \langle E_n(\Lambda_j \cap \Delta_n)h, h\rangle \quad \text{by Tonelli} \\ &= \sum_j \langle E(\Lambda_j)h, h\rangle. \end{aligned}$$

Hence, $E(\cup_j \Lambda_j) = \sum_j E(\Lambda_j)$. Therefore, E is a spectral measure.

Now, define $M := \int z \; dE(z)$ be defined as in Theorem 2. Our goal is to prove that $M = N$. We prove this by proving the two inclusions $M \subseteq N$ and $N \subseteq M$. Notice that $H_n \subseteq Dom(M)$ and the Spectral Theorem for bounded operators implies that for $h \in H_n$, $Mh = N_n h = Nh$. Since $H = \bigoplus_n H_n$, we may decompose an arbitrary $h \in H$ as $h = \sum_n h_n$ where $h_n \in H_n$ and $\sum_{n=1}^{\infty} ||Nh_n||^2 < \infty$. Moreover, since N is closed, $h \in Dom(N)$ and $Nh = Mh$. Therefore $M \subseteq N$. For the other inclusion, notice that Lemma 1 implies that M is closed. Therefore, by Proposition 1 it suffices to prove that $\{h \bigoplus Nh : h \in Dom(N^*N)\} \subseteq Gra(M)$ since this will prove that $Gra(M)$ is dense in $Gra(N)$.

Let $h \in Dom(N^*N)$. By our above work we know there exists a vector g such that $h = Bg$. Then

$$P_n Nh = P_n NBg = P_n Cg = C\; P_n g = NP_n h.$$

Thus, if $h_n := P_n h$ then $\sum_n ||Nh_n||^2 = \sum_n ||P_n Nh||^2 = ||Nh||^2 < \infty$ since $\sum_n P_n = 1$. So by definition we have $h \in Dom(M)$ and hence by the above argument $Nh = Mh$, i.e. $h \bigoplus Nh \in Gra(M)$. So, $M = N$ and hence $N = \int z \; dE(z)$, which proves (1).

To prove (2) and (3), we need to characterize the spectrum of N in terms of our bounded normal operators N_n.

Claim: $\sigma(N) = \overline{\bigcup_{n=1}^{\infty} \sigma(N_n)}$.

Indeed, if $\lambda \notin \overline{\bigcup_{n=1}^{\infty} \sigma(N_n)}$ then there is a $\delta > 0$ such that $|\lambda - z| \geq \delta$ for all $z \in \bigcup_{n=1}^{\infty} \sigma(N_n)$. Then it is clear that $(N_n - \lambda)^{-1}$ exists and $||(N_n - \lambda)^{-1}|| \leq \delta^{-1}$ for each n. Thus, the operator $A := \bigoplus_{n=1}^{\infty} (N_n - \lambda)^{-1}$ is a bounded operator. Since $H = \bigoplus_{n=1}^{\infty} H_n$ we have then that $A = (N - \lambda)^{-1}$ and hence $\lambda \notin \sigma(N)$. (converse). Hence the claim holds.

Now, if $\Delta \cap \sigma(N) = \emptyset$, then $\Delta \cap \sigma(N_n) = \emptyset$ for all n. Thus, $E_n(\Delta) = 0$ for all n and hence $E(\Delta) \leq \sum_n E_n(\Delta) = 0$ which proves (2).

If U is open and $U \cap \sigma(N) \neq \emptyset$, then the above claim shows that $U \cap \sigma(N_n) \neq \emptyset$ for some n_0. Since $E_{n_0}(U) \neq 0$, $E(U) \neq 0$ which proves (3).

Finally, let $A \in B(H)$ be such that $AN \subseteq NA$ and $AN^* \subseteq N^*A$. Then it follows that $A(1 + N^*N) \subseteq (1 + N^*N)A$ and hence $AB = BA$ by definition of B. By the Spectral Theorem for bounded operators, A therefore commutes with all the spectral projections of B. In particular this proves that H_n reduces A and if $A_n := A|_{H_n}$ then $A_n N_n = N_n A_n$. Therefore, $A_n E_n(\Delta) = E_n(\Delta) A_n$ for every Borel set $\Delta \subseteq \Delta_n$. By construction then it follows that $AE(\Delta) = E(\Delta)A$ for every Borel set Δ in $\mathbb{C}$. By considering simple functions and doing a standard density argument, part (4) of the theorem follows, which finishes the proof. $\square$

2 Application: Stone's Theorem

In this section, we use the Spectral Theorem to prove an important result which has implications to several areas of mathematics, including PDE's and mathematical quantum mechanics. As a motivating example, let A be a constant self-adjoint operator on an underlying Hilbert space H and consider the following ODE defined on $\mathbb{R} \times \mathbb{R}^+$:

$$\begin{aligned} i\partial_t \psi &= A\psi \\ \psi(0) &= \psi_0. \end{aligned}$$

This a simple example of the Schrödinger equation from quantum mechanics. From elementary ODE's, we know that given any $\psi_0 \in H$ the solution is given by

$$U(t)\psi_0 = e^{itA}\psi_0.$$

Moreover, this holds for more general self-adjoint operators A. Thus, it is important for the study of Schrödinger operators to understand properties of operators of the form e^{itA} where A is self-adjoint. We begin with the following theorem which summarizes the major properties of operators of this form.

Theorem 3. *If A is self-adjoint and $U(t) = e^{itA}$ for t in $\mathbb{R}$, then*

1. *U(t) is unitary;*
2. *U(s+t) = U(s)U(t) for all s in* $\mathbb{R}$*;*
3. *if $h \in H$, then* $\lim_{s \to t} U(s)h = U(t)h$*;*
4. *if $h \in Dom(A)$, then* $\lim_{t \to 0} \frac{1}{t}[U(t)h - h] = iAh$*;*
5. *if $h \in H$ and $\lim_{t \to 0} t^{-1}[U(t)h - h]$ exists, then $h \in Dom(A)$. Consequently, Dom(A) is invariant under each U(t).*

Proof. First, notice that since $\overline{e^{itx}}e^{itx} = e^{it(x-x)} = 1$ and $e^{itx}e^{isx} = e^{i(s+t)x}$, the first two parts follow directly from the functional calculus for normal operators. Also, since $e^{is(0)} = 1$, we have that $U(0)U(t) = U(t) = U(t)U(0)$, so that $U(0) = 1$.

Now, if $h \in H$, then using the group property from (2), we have

$$\begin{aligned} ||U(t)h - U(s)h|| &= ||U(t-s+s)h - U(s)h|| \\ &= ||U(s)[U(t-s)h - h]|| \\ &= ||U(t-s)h - h|| \end{aligned}$$

since $U(t)$ is unitary. Thus, it is sufficient to prove (3) when $t = 0$. Let $A = \int_{-\infty}^{\infty} x\, dE(x)$ be the spectral representation of A. Then

$$||U(t)h - h||^2 = \int_{-\infty}^{\infty} |e^{itx} - 1|^2 dE_{h,h}(x).$$

Now, since $E_{h,h}$ is a finite measure on $\mathbb{R}$, the facts that $\lim_{t\to 0} |e^{itx} - 1|^2 = 0$ and $|e^{itx} - 1|^2 \leq 4$ for each $x \in \mathbb{R}$ implies $U(t)h \to h$ as $t \to 0$ by the Lebesgue Dominated Convergence Theorem (LDCT). Thus, (3) is proved.

Note that if we define $f_t(x) = t^{-1}[e^{itx} - 1] - ix$, the functional calculus for normal operators implies that $f_t(A) = t^{-1}[U(t) - 1] - iA$. So if $h \in Dom(A)$,

$$\begin{aligned} \left\| \frac{1}{t}[U(t)h - h] - iAh \right\|^2 &= \|f_t(A)h\|^2 \\ &= \int_{-\infty}^{\infty} \left| \frac{e^{itx} - 1}{t} - ix \right|^2 dE_{h,h}(x). \end{aligned}$$

Now, $\lim_{t\to 0} f_t(x) = 0$ for all x in $\mathbb{R}$. Also, $|e^{is} - 1| \leq |s|$ for all real numbers s. Thus, $|f_t(x)| \leq |t|^{-1}|e^{itx} - 1| + |x| \leq 2|x|$. Since $|x| \in L^2(E_{h,h})$ by Theorem 2, the LDCT implies that (4) is true.

Finally, define $D = \{h \in H : \lim_{t\to 0} t^{-1}[U(t)h - h] \text{ exists in } H\}$. For h in D, let Bh be defined by

$$Bh = -i \lim_{t\to 0} \frac{U(t)h - h}{t}.$$

It is easy to see that D is a linear manifold in H and B is linear on D. Also, by (4), $B \supseteq A$ so that B is densely defined. Moreover, if $h, g \in D$, then

$$\langle Bh, g\rangle = -i \lim_{t\to 0} \left\langle \frac{U(t)h - h}{t}, g \right\rangle.$$

By (2) and the fact that each $U(t)$ is unitary, it follows that $U(t)^* = U(t)^{-1} = U(-t)$. Hence

$$\begin{aligned} \langle Bh, g\rangle &= -i \lim_{t\to 0} \left\langle h, \frac{U(-t)g - g}{t} \right\rangle \\ &= \lim_{t\to 0} \left\langle h, -i \left[\frac{U(-t)g - g}{-t} \right] \right\rangle \\ &= \langle h, Bg\rangle. \end{aligned}$$

Hence B is a symmetric extension of A. Since self-adjoint operators are maximal symmetric operators (from Section X.2 (2.11) from Conway), $B = A$ and $D = Dom(A)$. $\square$

We now turn our attention to proving a partial converse to the above theorem. Suppose we have a unitary group of operators $U(t)$ on a Hilbert space H. When does there exist a self-adjoint operator A such that $U(t) = e^{itA}$? It turns out that there is an extra condition which must be met. In particular, it is not always true that the conclusion of part (3) holds for an arbitrary unitary group. This motivates the following definition.

Definition 2. *A strongly continuous one parameter unitary group is a function $U : \mathbb{R} \to B(H)$ such that for all s and t in $\mathbb{R}$: (a) $U(t)$ is a unitary operator; (b) $U(s+t) = U(s)U(t)$; (c) if $h \in H$ and $t_0 \in \mathbb{R}$, then $U(t)h \to U(t_0)h$ as $t \to t_0$.*

Note that by above, if A is self-adjoint, then $U(t) = e^{itA}$ defines a strongly continuous one parameter unitary group. Also, $U(0) = 1$ and $U(-t) = U(t)^{-1}$, so that $\{U(t) : t \in \mathbb{R}\}$ is indeed a group.

Finally, we turn our attention to the proof of Stone's Theorem, which proves the partial converse to the above theorem. For motivation of the following proof, notice if $U(t) = e^{iAt}$ for some self-adjoint operator A, then we can recapture the generator A by computing the limit in part (4) of the above theorem. Most of the effort in the following proof will be in showing that a certain set (which will end up being the domain of our operator A) is dense in H. The rest of the proof is fairly straightforward: it is recommended that the reader omit the proof of the first claim on first reading.

Theorem 4 (Stone's Theorem)**.** *If U is a strongly continuous one parameter unitary group of operators, then there exists a self-adjoint operator A such that $U(t) = e^{iAt}$.*

Proof. First, define $\mathcal{D} := \{h \in H : \lim_{t\to 0} t^{-1}(U(t)h - h) \text{ exists}\}$. Notice that $\mathcal{D} \neq \emptyset$ since $0 \in \mathcal{D}$. Moreover, $\mathcal{D}$ is clearly a linear manifold of H. By our above work, we expect $\mathcal{D}$ to essentially be the domain of our self-adjoint operator A. Thus the next claim should be expected.

Claim: $\mathcal{D}$ is dense in H.

Indeed, let $\mathcal{L} := \{\phi \in C(\mathbb{R}) : \phi \in L^1(0, \infty)\}$. We will use $\mathcal{L}$ to construct a subset of $\mathcal{D}$ which is dense in H by the strong continuity of the group U. The proof of this fact is technical and should probably be omitted upon first reading.

Notice that given any $\phi \in \mathcal{L}$ and $h \in H$, the function $t \to \phi(t)U(t)h$ is a continuous function from $\mathbb{R}$ to H. By the definition of $\mathcal{L}$ and the fact that $||U(t)h|| = ||h||$ for all $h \in H$ and all $t \in \mathbb{R}$, the Riemann integral $\int_0^\infty \phi(t)U(t)h\, dt$ can be defined and is clearly a vector in H. Define the operator $T_\phi : H \to H$ by

$$T_\phi h := \int_0^\infty \phi(t)U(t)h\, dt.$$

Clearly, $||T_\phi|| \leq \int_0^\infty |\phi(t)|dt$ so that T_ϕ is a bounded linear operator on H. Similarly, for each $\phi \in \mathcal{L}$, the operator $S_\phi : H \to H$ defined by

$$S_\phi h := \int_0^\infty \phi(t)U(-t)h\, dt$$

isa bounded linear operator on H.

Now, since $U(t+s) = U(t)U(s)$ for all $s, t \in \mathbb{R}$, we see that

$$\begin{aligned} U(t)T_\phi h &= \int_0^\infty \phi(s)U(t+s)h\, ds \\ &= \int_t^\infty \phi(s-t)U(s)h\, ds \end{aligned}$$

and similarly

$$U(t)S_\phi h = \int_{-t}^{\infty} \phi(s+t)U(-s)h\ ds.$$

Define $\mathcal{L}' := \{\phi \in \mathcal{L} \cap C^1(\mathbb{R}) : \phi' \in \mathcal{L}\}$. Then for each $\phi \in \mathcal{L}'$ we have

$$\begin{aligned} \frac{-i}{t}\left(U(t)-1\right)T_\phi h &= \frac{-i}{t}\int_t^\infty \phi(s-t)U(s)h\ ds + \frac{i}{t}\int_0^\infty \phi(s)U(s)h\ ds \\ &= -i\int_t^\infty \left(\frac{\phi(s-t)-\phi(s)}{t}\right)U(s)h\ ds + \frac{i}{t}\int_0^t \phi(s)U(s)h\ ds \end{aligned}$$

Since $||U(t)h|| = ||h||$, the triangle inequality implies that

$$\left\|\int_0^t \left(\frac{\phi(s-t)-\phi(s)}{t}\right)U(s)h\ ds\right\| \le ||h|| \sup_{0\le s\le 1} |\phi(s-t)-\phi(s)|$$

which goes to 0 as $t \to 0$. Therefore,

$$\begin{aligned} \lim_{t\to 0}\int_t^\infty \left(\frac{\phi(s-t)-\phi(s)}{t}\right)U(s)h\ ds &= -\int_0^\infty \phi'(s)U(s)h\ ds \\ &= -T_{\phi'}h. \end{aligned}$$

Moreover, the fact that the function $t \to U(t)h$ is continuous and that $U(0) = 1$ yields

$$\lim_{t\to 0}\frac{1}{t}\int_0^t \phi(s)U(s)h\ ds = \phi(0)h$$

by the Fundamental Theorem of Calculus. Therefore, for all $\phi \in \mathcal{L}'$ and $h \in H$ we have that

$$\lim_{t\to 0}\frac{-i}{t}(U(t)-1)T_\phi h = iT_{\phi'}h + i\phi(0)h.$$

Hence, we have that $\{T_\phi h : \phi \in \mathcal{L}' \text{ and } h \in H\} \subseteq \mathcal{D}$.

Now, we claim that for each $n \in \mathbb{N}$ there exists $\phi_n \in \mathcal{L}'$ such that $\phi_n \ge 0$, $\phi_n(t) = 0$ for $t \ge \frac{1}{n}$, and $\int_0^\infty \phi_n(t)dt = 1$. Indeed, since $\left[\frac{1}{2}, \frac{3}{4}\right]$ is a compact subset of $(0,1)$, there exists a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi \ge 0$, $\psi(x) > 1$ on $\left[\frac{1}{2}, \frac{3}{4}\right]$, $\psi(x) = 0$ for $x \notin (0,1)$, and $\int_{\mathbb{R}} \psi(x)dx = 1$. Define $\phi_n(x) = n\psi(nx)$ and notice that ϕ_n clearly satisfies the desired conditions. Hence,

$$T_{\phi_n}h - h = \int_0^{1/n} \phi_n(t)(U(t)-1)h\ dt$$

so that $\|T_{\phi_n}h - h\| \le \sup_{0\le t\le 1/n} \|U(t)h - h\|$. Since the group U is strongly continuous, it follows that $\|T_{\phi_n}h - h\| \to 0$ as $n \to \infty$ for all $h \in H$. Therefore, $\mathcal{D}$ is dense in H as claimed.

Now, for $h \in \mathcal{D}$ we continue with our intuition and define

$$Ah := -i\lim_{t\to 0}\frac{(U(t)-1)h}{t}.$$

The above operator is thus a densely defined linear operator, and we expect that this A is the infinitesimal generator for the group U. As a first step in showing that A is self-adjoint, we make the following claim.

Claim: A is symmetric.

Indeed, let $h, g \in \mathcal{D}$ and notice that

$$\begin{aligned}
\langle Ah, g\rangle &= -i \lim_{t\to 0} t^{-1} \left(\langle U(t)h, g\rangle - \langle h, g\rangle\right) \\
&= -i \lim_{t\to 0} (t)^{-1} \left(\langle h, U(-t)g\rangle - \langle h, g\rangle\right) \\
&= -i \lim_{t\to 0} (-t)^{-1} \left(\langle h, U(t)g\rangle - \langle h, g\rangle\right) \\
&= \langle h, Ag\rangle
\end{aligned}$$

as claimed.

Now, since $A \subseteq A^*$ it is clear that A is closable. By a slight abuse of notation, we denote the unique closure of A by A. In order to prove that A is in fact self-adjoint, it suffices to prove that $Ran(A \pm i)$ is dense in H, since this clearly implies that $ker(A^* \pm i) = \{0\}$. Proceeding as above, notice that

$$(A + i)T_\phi = AT_\phi + iT_\phi = i(T_{\phi'} + T_\phi) + i\phi(0)$$

for all $\phi \in \mathcal{L}'$. Taking $\phi(t) = -ie^{-1}$ it follows that $(A + i)T_\phi = 1$. Similarly, one has that if $\phi(t) = ie^{-1}$, then $(A - i)S_\phi = 1$. Hence, the operator A is indeed self-adjoint.

For the last step in the proof, we need to prove that if we define $V = e^{itA}$, then $V = U$. To this end, let $h \in \mathcal{D}$ and notice that V is a strongly continuous unitary group of operators. Hence

$$s^{-1}(V(t + s) - V(t))h = s^{-1}(V(s) - 1)V(t)h \to iAV(t)h$$

and hence $V'(t)h = iAV(t)h$. Similarly, it follows that $U'(t)h = iAU(t)h$. Define $h(t) = U(t)h - V(t)h$ and note that $h : \mathbb{R} \to H$ is differentiable by above. Moreover, $h'(t) = iA(U(t) - V(t))h = iAh(t)$. Furthermore, since A is self-adjoint we have

$$\begin{aligned}
\frac{d}{dt}\|h(t)\|^2 &= \langle h'(t), h(t)\rangle + \langle h(t), h'(t)\rangle \\
&= \langle iAh(t), h(t)\rangle + \langle h(t), iAh(t)\rangle \\
&= 0
\end{aligned}$$

and hence $\|h\| : \mathbb{R} \to \mathbb{R}$ is a constant function. Since $h(0) = 0$, it follows that $V(t)h = U(t)h$ for all $h \in \mathcal{D}$. Since $\mathcal{D}$ is dense in H, it follows that $U = V$ as desired. $\square$