VI News in dimension 3 and higher

General fact: Every linear map $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$
is uniquely determined by
a matrix $a = (a_{ij})$ such that

$$T(e_i) \boxed{= \cancel{(a_{i1}, \dots \quad a_{in})}} = (a_{1i}, \dots, a_{ni})$$

also works but changes matrix mult.

Here (e_i) are the standard unit vectors

$$e_i = (0 - - \underset{i\text{-th position}}{1} - \cdot\ )$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1i} \\ \\ a_{mi} \end{pmatrix}$$ better way to remember

Comment: The choice of a concrete basis in $\mathbb{R}^n$
is not canonical

(à la Einstein: Physics laws are not allowed to depend on it!)

Recall $b_1 \ldots b_n \in \mathbb{R}^n$ basis iff

$U(e_i) = b_i$ extends to a linear isomorphism on $\mathbb{R}^n$

(linear isomorphism $\Leftrightarrow$ linear and there is a map $V: \mathbb{R}^n \to \mathbb{R}^n$ such that $V(b_i) = e_i$)
linear

(The last fact is obvious because e_i is also a basis)

Fact 3 the new matrix of T with basis (b_i) is given by $u^{-1} a u$

where u is the matrix of transformation $U(e_i) = b_i$

Here ab is the composition of matrices or matrix multiplication.

Fact 2 $T: \mathbb{R}^n \to \mathbb{R}^n$ has matrix a

$S: \mathbb{R}^n \to \mathbb{R}^n$ — " — b

Then ST has matrix ba.

Proof of Fact 2 ($\rightarrow$ also proof of Fact 3)

$$T(e_i) = \sum a_{ij} e_j$$

$$\begin{aligned}
ST(e_i) &= S\left(\sum a_{ij} e_j\right) \\
&= \sum_j a_{ij}\, S(e_j) \\
&= \sum_j a_{ij} \sum_k b_{jk} e_k \\
&= \sum_k \left(b_{jk} \sum_j a_{ij}\right) e_k \\
&= \sum_k \left(\sum_j a_{ij} b_{jk}\right) e_k
\end{aligned}$$

Proof of Fact 2 ($\leadsto$ also proof of Fact 3)

$$T(e_i) = \sum_j a_{ji} e_j$$

$$ST(e_i) = \sum_j a_{ji} S(e_j)$$

$$= \sum_j a_{ji} \sum_k b_{kj} e_k$$

$$= \sum_k \left(\sum_j a_{ji} b_{kj}\right) e_k$$

$$= \sum_k \left(\sum_j b_{kj} a_{ji}\right) e_k \qquad \square$$

Fact IV T with matrix a T is a linear isomorphism

if and only if

$\det a \neq 0$. Moreover $\det ab = \det a \det b$

Here

$$\det a = \sum_{\sigma \in \mathrm{Perm}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

$\mathrm{sgn}(\sigma)$: σ can be written as a product of inversions

(ij) inversion

$\sigma = (i_1 j_1)(i_2 j_2) \dots (i_m j_m)$ $\quad \mathrm{sgn}(\sigma) = \begin{cases} 1 & m \text{ even} \\ -1 & m \text{ odd.} \end{cases}$

We only need determinants in dimension 3

$$\det a = \left| a \right| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{33}$$

$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

(Let us skip the proof, because we want to do calculus after all).

Note

$$\det a = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\substack{\text{erase 1 column} \\ \text{1 row}}} - a_{21}\underbrace{\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}}_{\substack{\text{erase 1 colum} \\ \text{2 row}}} + a_{31}\underbrace{\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}}_{\substack{\text{erase 1 col} \\ \text{3 row}}}$$

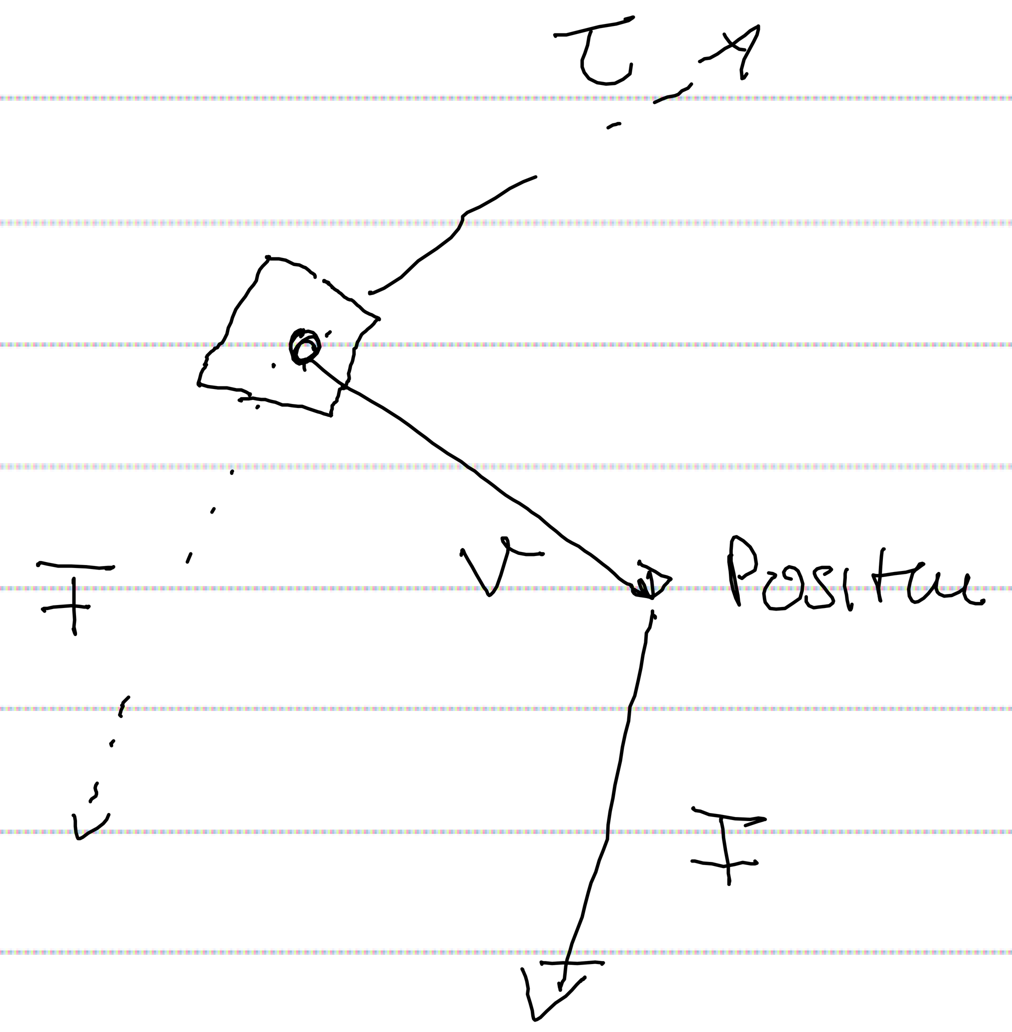
erase i-th colum k-row

$$= \sum_k a_{1k}(-1)^{1+k} \det A_{1k}$$ (several ways)

VII A speciality in dimension 3

(⟶ section 12.4)

Physically : Torque



F force vector
v position vector

Then we get rotation around $\tau = v \times F$

Direction of τ rotation axis

magnitude of τ how much rotation

Or How do we screw things up?

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = e_1(a_2b_3 - a_3b_2) - e_2(a_1b_3 - a_3b_1) + e_3(a_1b_2 - a_2b_1)$$

$$= (a_2b_3 - a_3b_2, -a_1b_3 + a_3b_1, a_1b_2 - a_2b_1)$$

Note

$$(c, a \times b) = \sum_j c_j (a \times b)_j$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Therefore $a \times b$ is orthogonal to any vector in $V = \langle a, b \rangle$ plane !

(recall $\det = 0$ if linear dependent)

Note $|a \times b| = ?$

Observation : This is a coordinate free object. ~~as~~ — we may move vectors with transformations not changing inner product.

Example

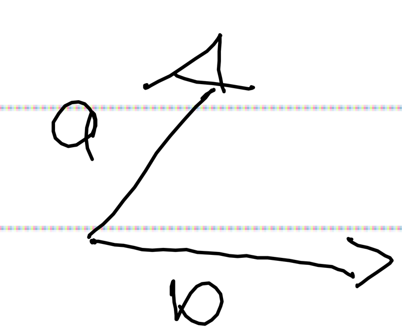
$$a = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}$$

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ s & 0 & 0 \\ \alpha & \beta & 0 \end{vmatrix} = 0 \cdot e_1 - 0 \cdot e_2 + s\beta e_3 = (0, 0, s\beta)$$

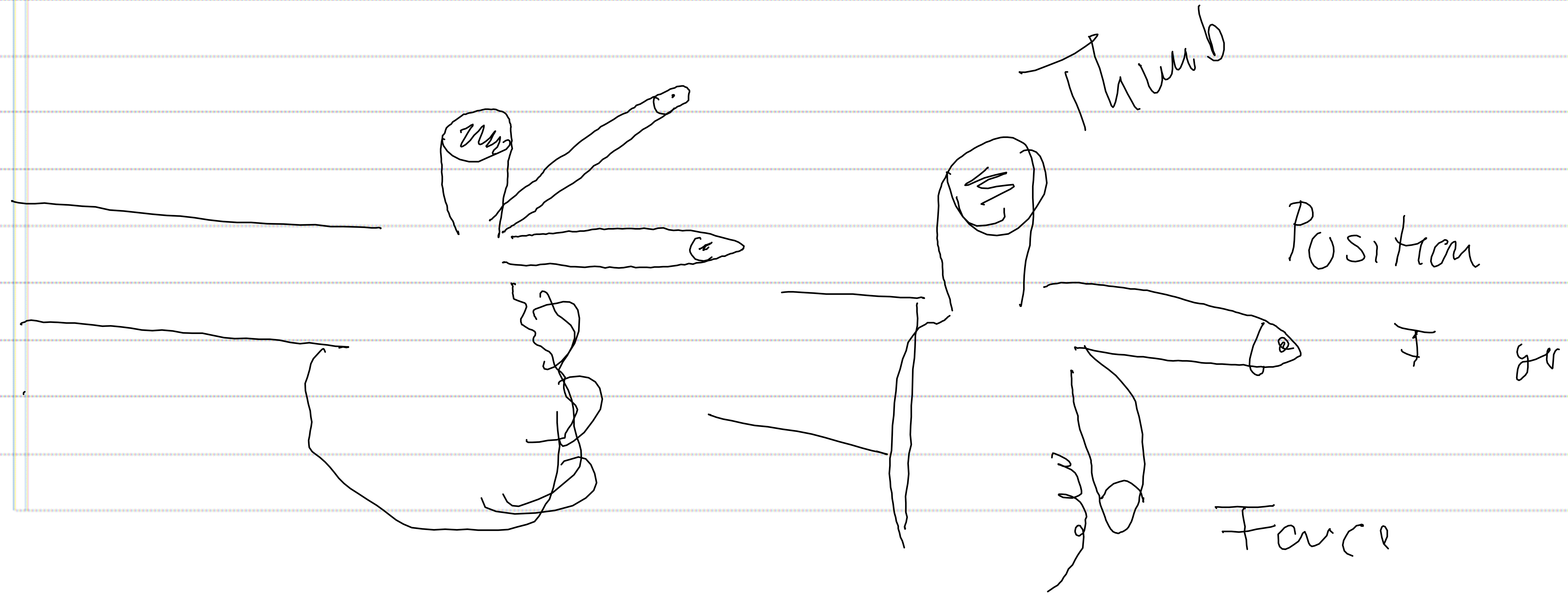
$$|a \times b| = |s|\,|\beta| = |a|\,|b| \frac{|\beta|}{\sqrt{\alpha^2 + \beta^2}}$$

$$= |a|\,|b|\,|\sin\theta|$$

$= \text{area of } \langle a, b \rangle$



Finally: Orientation follows right hand rule

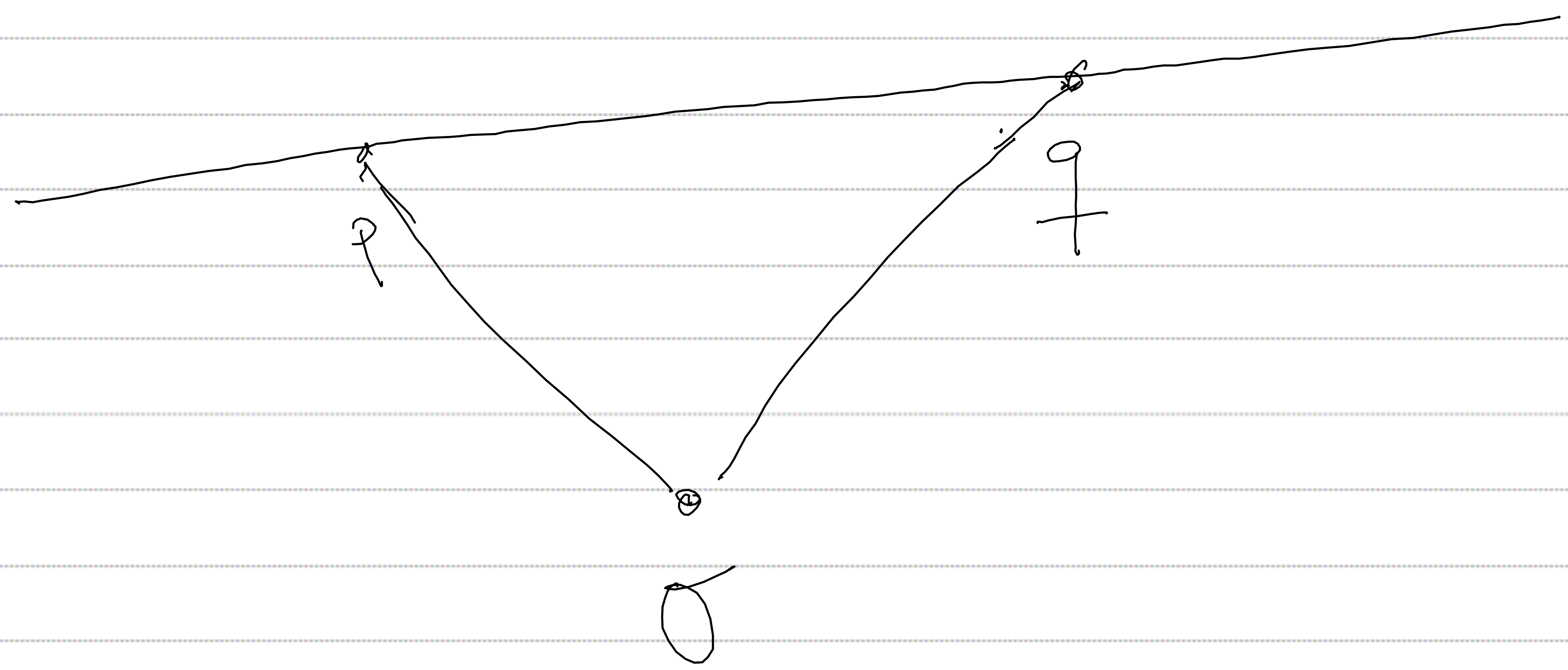


$\overline{VII}$ Equations for Line, Planes, etc

Line is given by

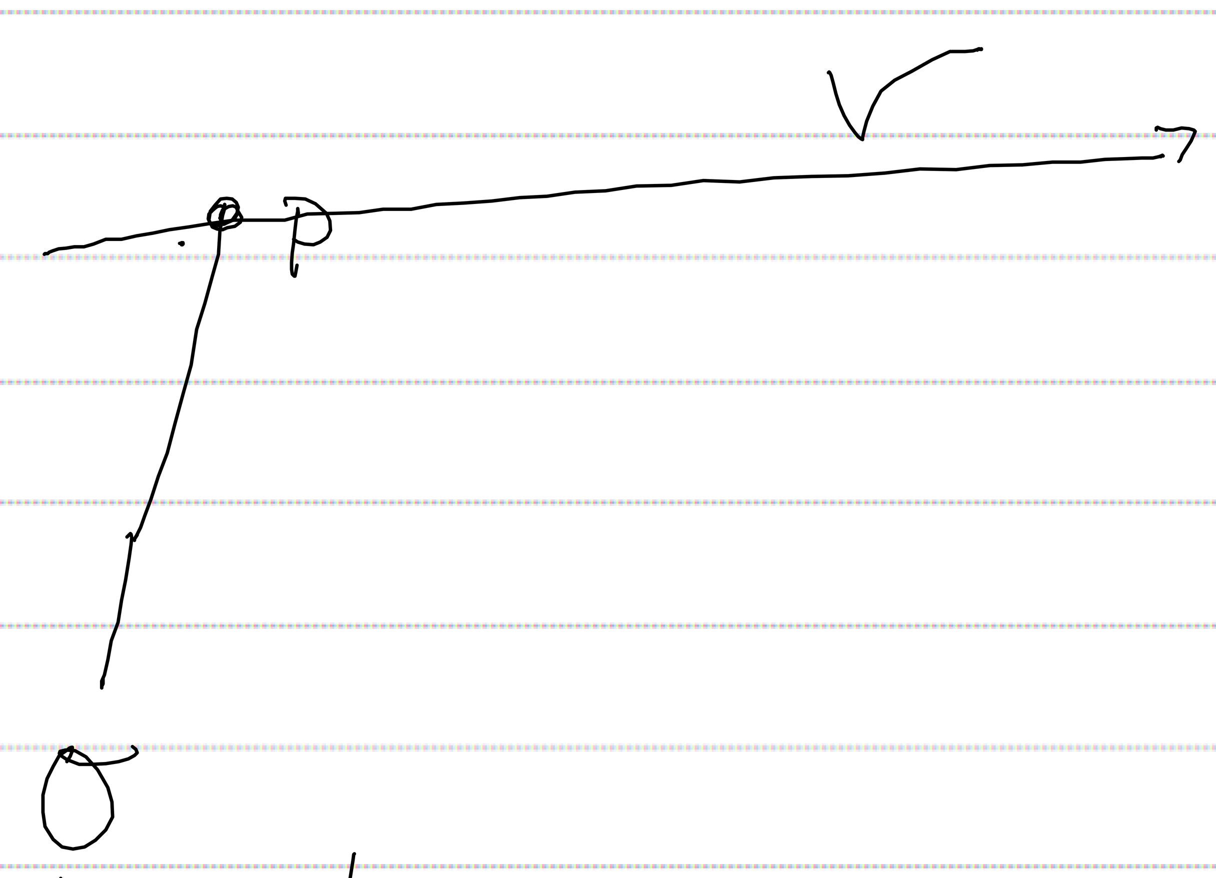
1). two points $p = (p_1\, p_2\, p_3)$

$q = (q_1\, q_2\, q_3)$



$l = ?$

or 2). a point



and a direction v

$l = \{ p + tv \, : \, t \in \mathbb{R} \}$

or 3). an equation

Description 2) allows us to calculate 3

$p = (p_1\ p_2\ p_3) \qquad V = (V_1\ V_2\ V_3)$

$$\ell = \left\{ \begin{pmatrix} p_1 + tV_1 \\ p_2 + tV_2 \\ p_3 + tV_3 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$x_1 = p_1 + tV_1 \qquad x_1 - x_2 = (p_1 - p_2) + t(V_1 - V_2)$$
$$x_2 = p_2 + tV_2 \qquad x_3 - x_2 = (p_3 - p_2) + t(V_3 - V_2)$$
$$x_3 = p_3 + tV_3$$

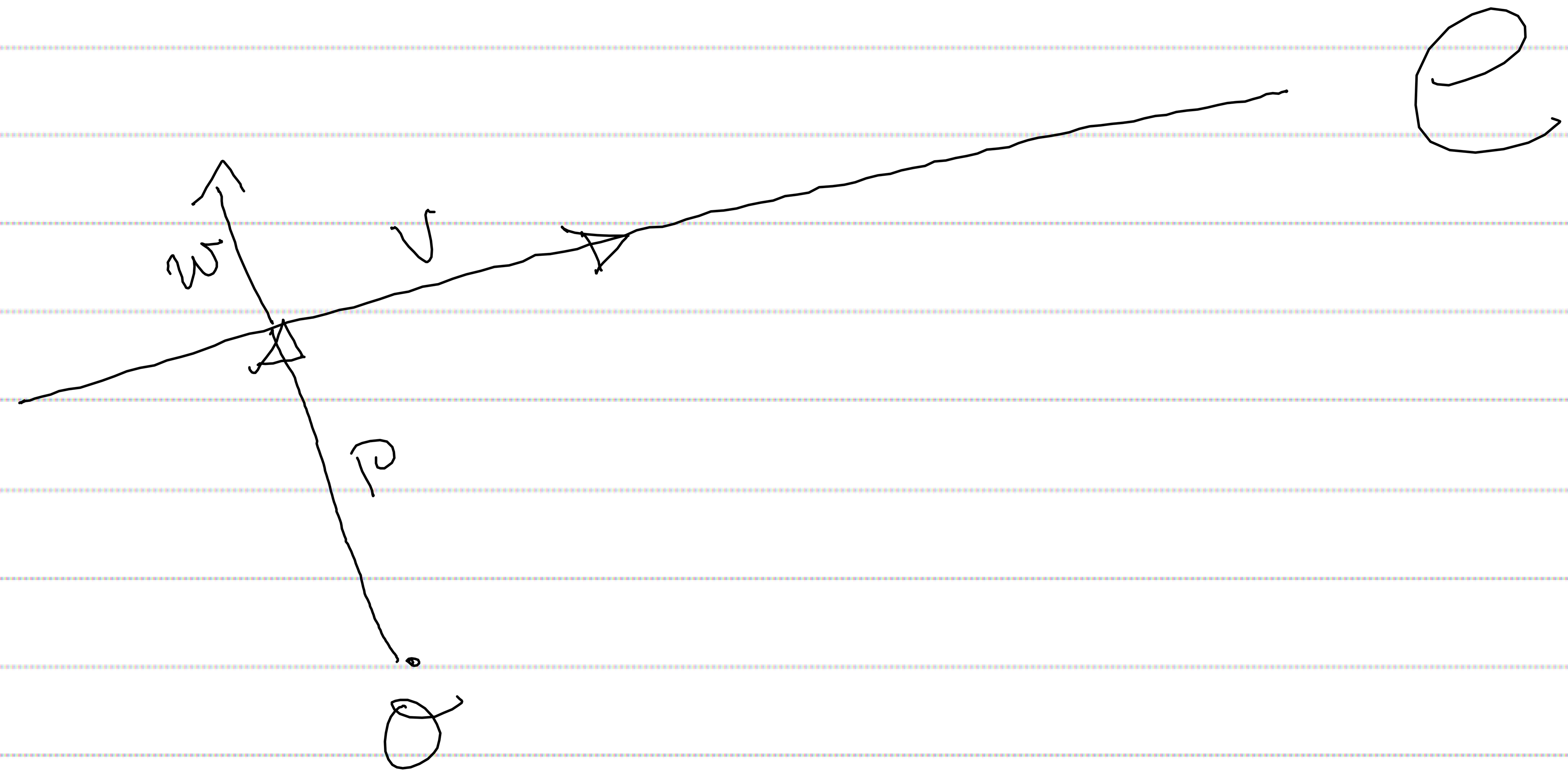
Somehow need two equations

Note throug origin

$$\ell = \left\{ \begin{pmatrix} tx_1 \\ tx_2 \\ tx_3 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Not through origin

Preparational detour $\mathbb{R}^2$



Let w be orthogonal to v $\quad (w,v)=0$

choose p on ℓ with distance from origin minimal.

Then $\ell = \{ z \in \mathbb{R}^2 : (z,w) = (p,w) \}$

Example a) $\ell = \{ (x,0) : x \in \mathbb{R} \}$

$= \{ (z_1, z_2) : ((z_1,z_2),(0,1)) = 0 \}$

b) $\ell = \{ (x,3) : x \in \mathbb{R} \}$

$= \{ (z_1,z_2) : (z,(0,1)) = 3$

c) $\ell = \{ (x+1, 2x+3) : x \in \mathbb{R} \}$

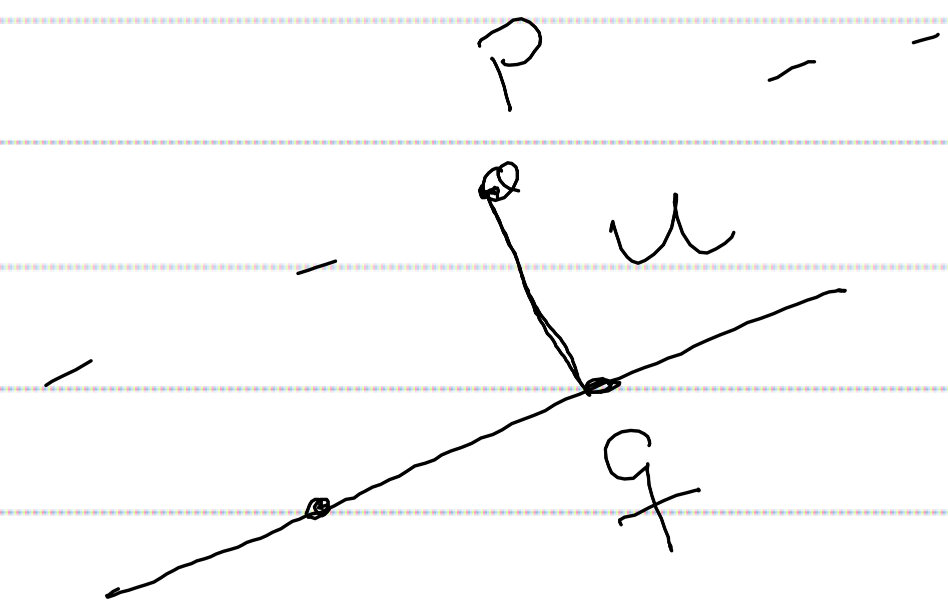
$\ell_0 = \{ (x, 2x) : x \in \mathbb{R} \}$ $\quad \ell = (1,3) + \ell_0$

Iguae

$\mathrm{dist}((1,3), \ell_0)$

$\ell_0 = V_0$ 1 dimensional subspace

$b_0 = \frac{(1,2)}{\sqrt{3}}$ unit vector



best approximation

$w_0 = ((1,3), b_0) b_0 = \frac{1}{3}((13),(12))\,(12)$

$= \frac{7}{3}(1,2)$

$u_0 = (13) - w_0 = (13) - \frac{7}{3}(1,2)$

$= \left(-\frac{4}{3}, -\frac{5}{3}\right)$

$9-14$

$= \frac{19}{3}$

$\left((13), \left(\frac{4}{3}, \frac{5}{3}\right)\right)$

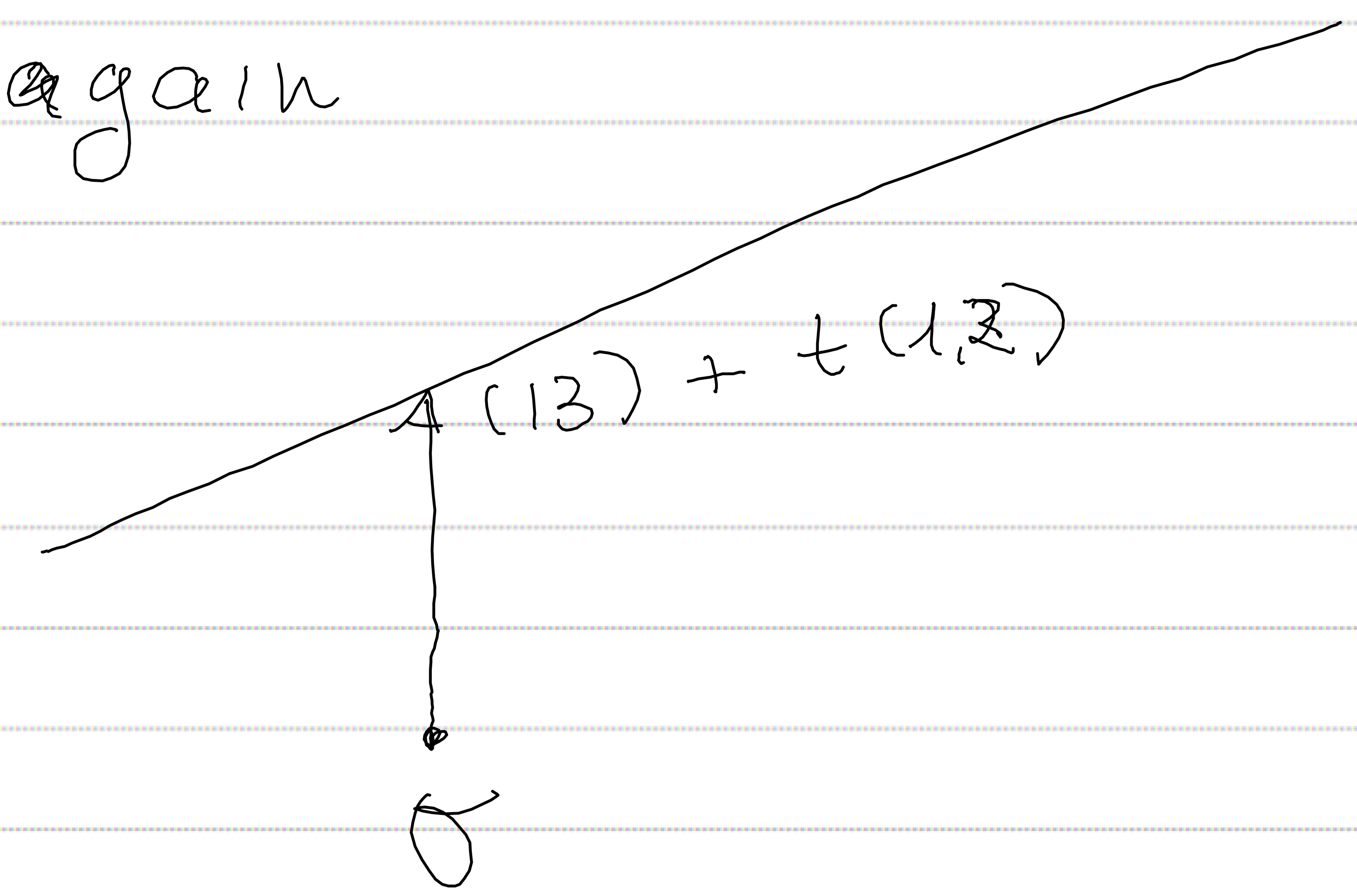
$\ell = \{ z \in \mathbb{R}^2 : (z,u) = ((1,3),u) \}$?

Indeed $= \{(z_1,z_2) : \frac{4}{3}z_1 + \frac{5}{3}z_2 = \frac{19}{3}\}$

$= \{(z_1,z_2) : 4z_1 + 5z_2 = 19\}$

$z = (1,3) + t(1,2)$ $\quad (z,u) = ((13),u) + t((12)\,u)$

again



Need u orthogonal to v

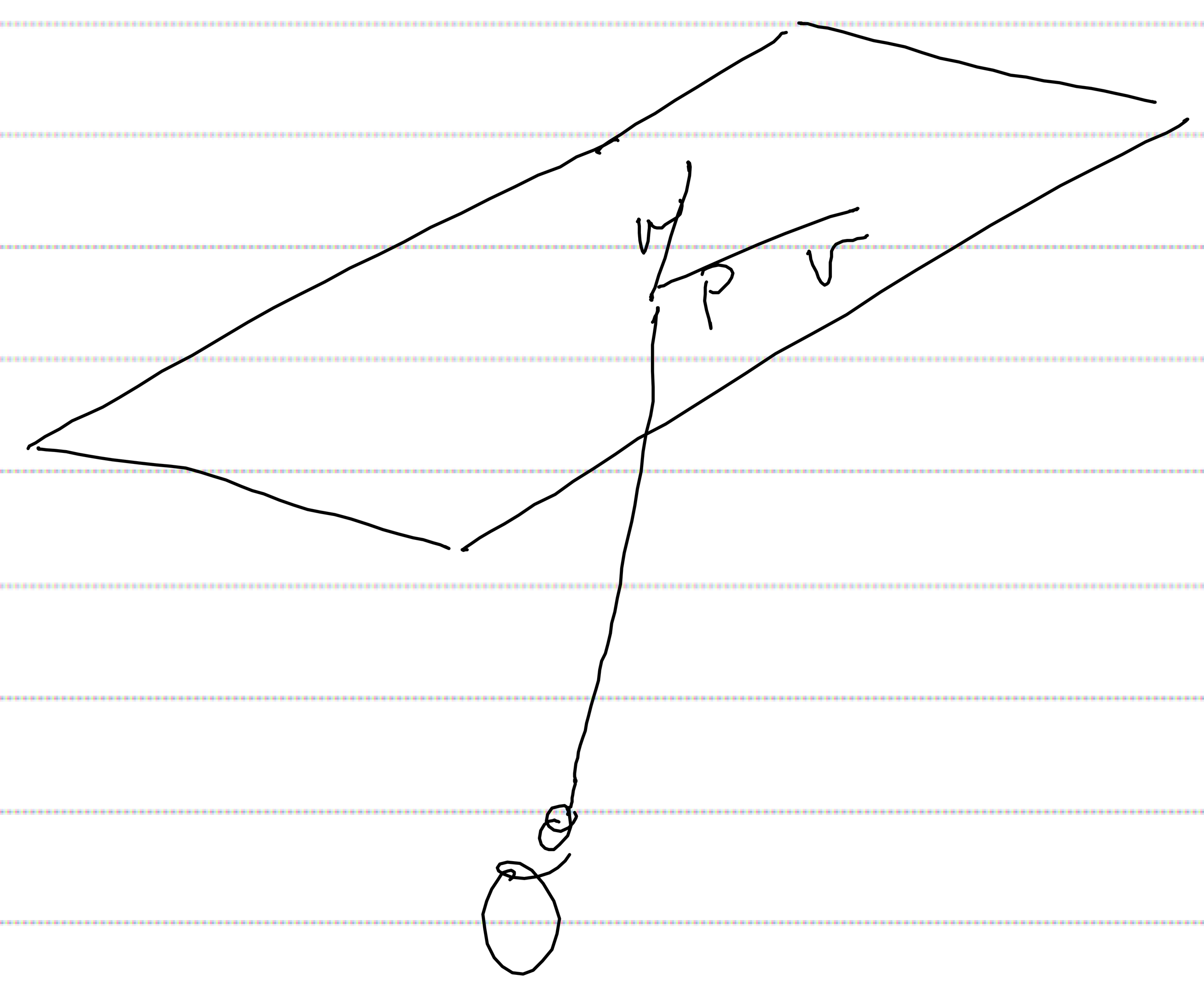
$$u = (-2, 1)$$

$$(u, p + tv) = (u, p) + t(u, v) = (u, p)$$

$$((-2,1),(x,y)) = (u,p) = ((-2,1),(13)) = 1$$
$$= -2 + 3$$

Equation $\boxed{-2x + y = 1}$

Plane equation in $\mathbb{R}^3$



given by one
2 vectors and
one point

$$P = \{ p + tv + sw : t, s \in \mathbb{R} \}$$

$$= \left\{ \begin{pmatrix} p_1 + tv_1 + sw_1 \\ p_2 + tv_2 + sw_2 \\ p_3 + tv_2 + sw_3 \end{pmatrix} : t, s \in \mathbb{R}^2 \right\}$$

$$\boxed{\text{Parametization of Plane (and there are many)}}$$

Equation Find $c = v \times w$ which is perpendicular and then

plane

$P = \{(x, y, z) : c_1 x + c_2 y + c_3 z = d\}$

$d = (c, P)$

point

Equation for plane

Line = Intersection of two plane

$\hat{=}$ two equations

Parametric ~~equation~~

$\ell = \{P_0 + tv : t \in \mathbb{R}\}$

$= \{(x, y, z) : ((x, y, z), w) = \alpha$
$((x, y, z), \tilde{w}) = \beta\}$

$(w, v) = 0 \quad (\tilde{w}, v) = 0$

$(P_0, w) = \alpha \quad (P_0, \tilde{w}) = \beta.$

Example

$$\ell = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} , \quad t \in \mathbb{R} \right\}$$

$$\big((1,1,0) , (1 -1 \ 1) \big) = 0$$

$$c = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 1 \cdot e_1 \cancel{+} 1 e_2 + 2 e_3$$

$$= (-1 \ 1 \ 2)$$

$$\alpha = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) = 2$$

$$\beta = \left(\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) = -1 + 1 + 2 = 2$$

$$\ell = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{array}{l} x + y = 2 \\ -x + y + 2z = 2 \end{array} \right\}$$

$\uparrow$

Equation

Short version

Plane equation

$$\boxed{ax+by+cz=d} \qquad \text{in } \mathbb{R}^3$$

or

or $\boxed{\text{parametization}}$

$$P=\left\{\begin{pmatrix} p_1+tv_1+sw_1 \\ p_2+tv_2+sw_2 \\ p_3+tv_3+sw_3 \end{pmatrix} : s,t\in\mathbb{R}^2\right\}$$

(Any other basis for $V=\langle v,w\rangle$ the span)

or three points (generically)

Line equation

$$a_1x+b_1y+c_1z=d_1$$

$$a_2x+b_2y+c_2z=d_2$$

or

$$\ell=\left\{\begin{pmatrix} p_1+tv_1 \\ p_2+tv_2 \\ p_3+tv_3 \end{pmatrix} : t\in\mathbb{R}\right\}$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}\in\ell \text{ and } \begin{pmatrix} p_1+v_1 \\ p_2+v_2 \\ p_3+v_3 \end{pmatrix}\in\ell$$

two points

HW given three points find equation.

Note In $\mathbb{R}^2$ lines are given by one equation only.

VIII Equations for certain geometric objects

a) Circle or Sphere

$|p - z| = r$ p midpoint

$z = (x_1, x_2, x_3)$ r distance

variable

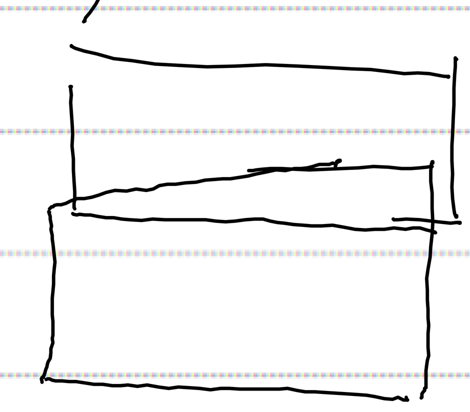
b) Distance between points

$|\overrightarrow{P_1 P_2}| = dist(P_1, P_2) = |\underbrace{P_2 - P_1}_{\text{vector}}|$

c) Distance from a plane

$P = \{ p_0 + tv + sw \; : \; t, s \in \mathbb{R} \}$

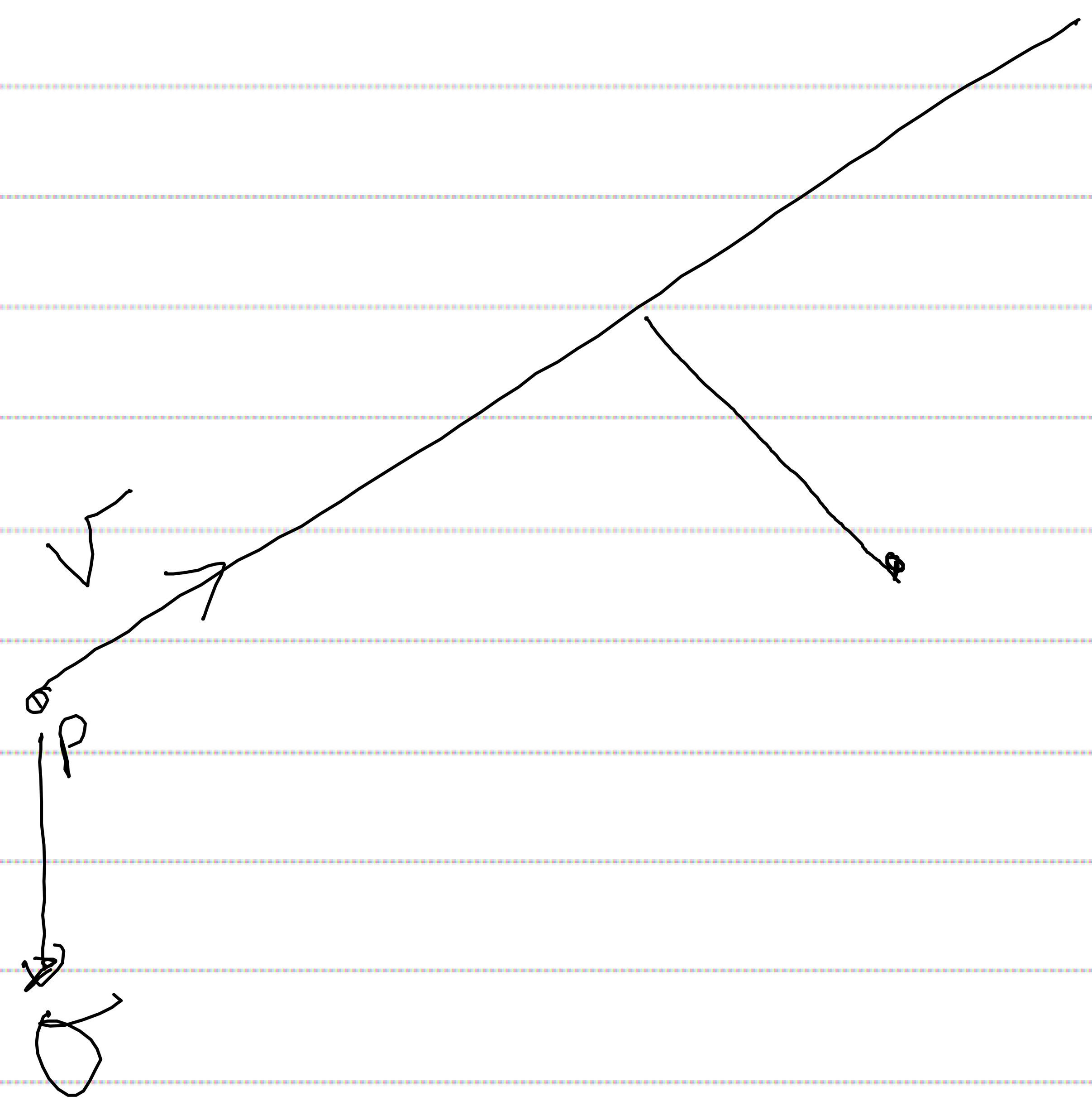
~~Assume~~ We know $c = \dfrac{v \times w}{|v \times w|}$ is orthogonal to v, w



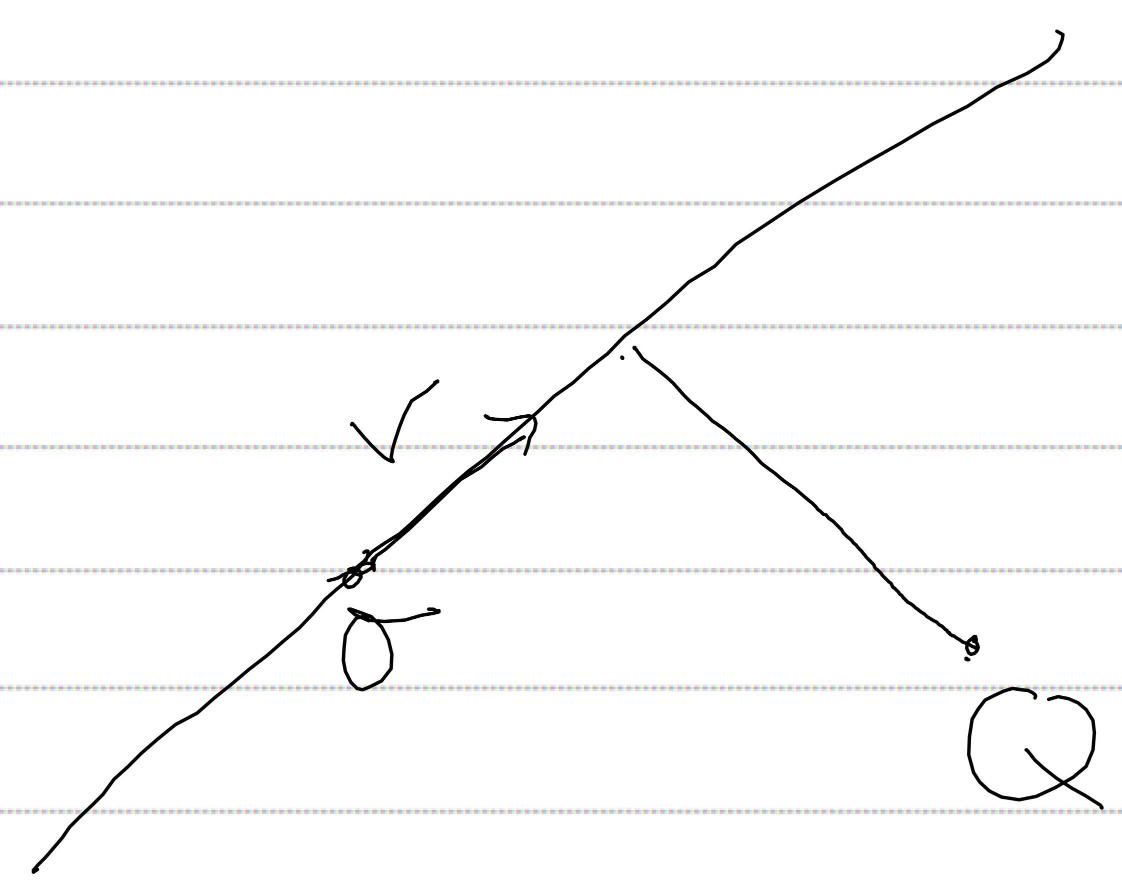
$p_0 = 0$ $|(c, z)| =$ distance

$p_0 \neq 0$ $|(c, z - p_0)| =$ distance

d) distance forom a line



through origin



$|v| = 1$

best approximation $(Q, v)v$

Distance $|Q - (Q,v)v|$

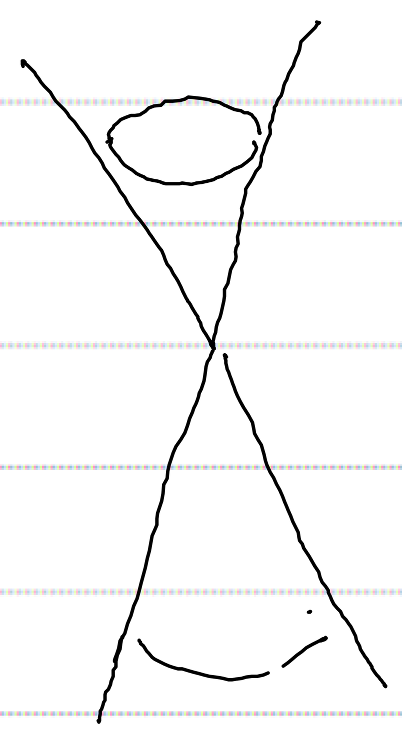
$$= \sqrt{|(Q, b_2)|^2 + |(Q, b_2)|^2}$$

normal Indeed

where b_2, b_3 are orthognal vectors so that (v, b_2, b_3) is a basis

Not through origin take difference.

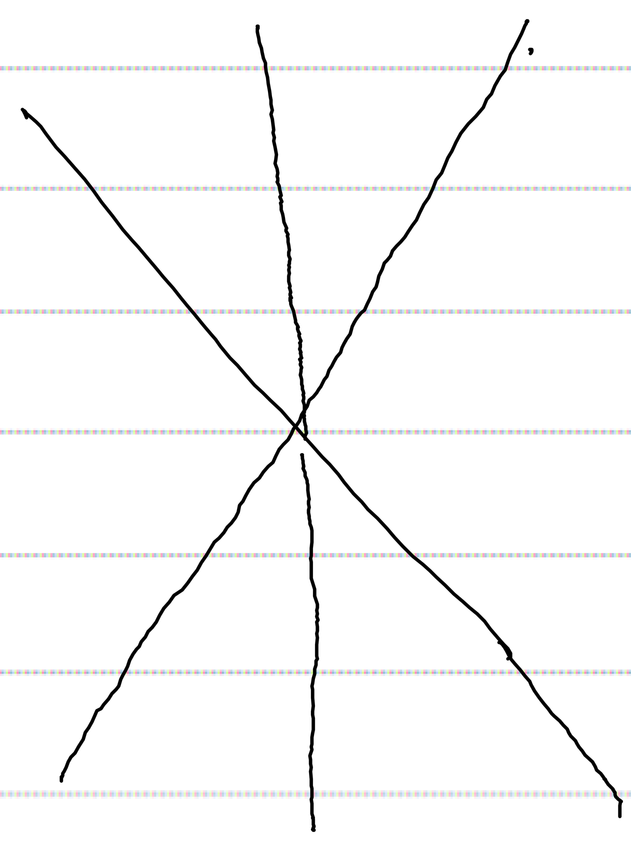
e) Very popular for old greeks



cone

Equation ?

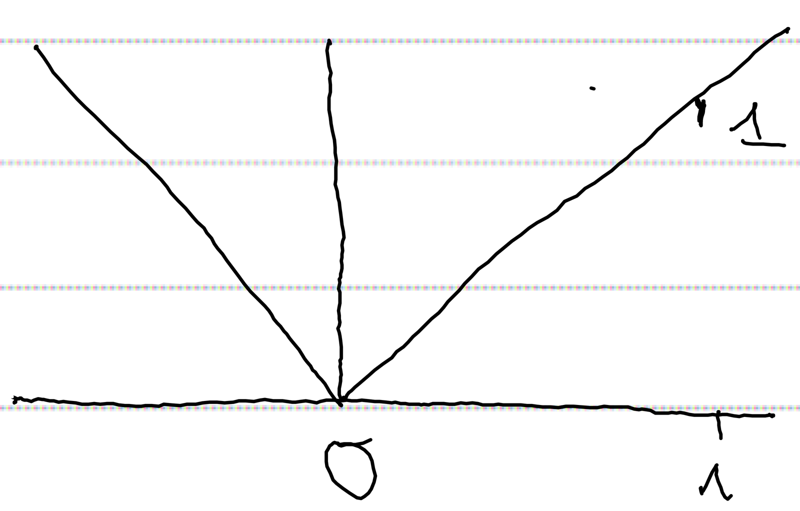
f) in $\mathbb{R}^2$



$y^2 = \alpha x^2$

$\alpha > 0$
parameter

What is this function ?



$y = f(x) = |x|$

Hence in $\mathbb{R}^3$

$$y^2 + z^2 = \alpha x^2$$ cone equation

Change of basis $$\boxed{\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x^2}{a^2}}$$

g) What are the possible intersections between a cone and a plane

- circle — parallel to yz plane
- ellipse — a little tilted

- parabola

- hyperpola

Problem: Find equations!

Hint: plane $P = \left\{ \begin{pmatrix} p_1 + tv_1 + sw_1 \\ p_2 + tv_2 + sw_2 \\ p_3 + tv_3 + sw_2 \end{pmatrix} : s,t \in \mathbb{R}^2 \right\}$

and
$$\frac{(p_1 + tv_1 + sw_1)^2}{a^2} = \frac{(p_2 + tv_2 + sw_2)^2}{b^2} + \frac{(p_3 + tv_3 + sw_3)^2}{c^2}$$

or
$$\frac{p_1^2 + t^2 v_1^2 + s^2 w_1^2 + 2tp_1v_1 + 2sp_1w_1 + 2ts\, v_1 w_2}{a^2} = \dots$$

$\boxed{\text{This are just quadratic equations in } \mathbb{R}^2}$

h) Quadratic equations in $\mathbb{R}^3$

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \pm \frac{z^2}{c^2} + d$$ (no x, y, z terms)

h1) $\cancel{\pm}$ + $\frac{x^2}{a} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = d$ ellipsoid

$\parallel$

transferred ball (sphere)

h2) $\frac{+}{+}$ hyperboloid

h3) $\frac{-}{+}$ $\frac{x^2}{a^2} = \frac{y^2}{b^2} + \frac{z^2}{c^2}$ cone

h4) hyperbolic paraboloid $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Remark general quadratic equation

$$ax^2 + by^2 + cz^2 + \alpha xy + \beta yz + \gamma xz$$
$$+ \rho x + \nu y + \mu z + \kappa = 0$$

Through clever change of variable one can simplify to the cases mentioned above