Distance from a subspace

Let V be an inner product space

$|v| = (v|v)^{1/2}$ length of a vector

Distance $\quad p, q \in V \quad \text{dist}(p,q) = |p-q|$

Pb $\quad p \in V \quad W \subseteq V$ be a subspace (of finite dimension)

Find $\quad w \in W$ such that

a) $\quad \min_{\tilde{w}} |\tilde{w} - p| = |w - p|$

b) $\quad |w - p|$

Solution Choose an orthonormal basis $b_1 \ldots b_k$ of W

Define

$$w = \sum_k (b_k|p)\, b_k$$

Define $z = p - v$

$$= \begin{cases} 1 & j = k \\ 0 & \text{else} \end{cases}$$

Claim a) $\quad (z|b_j) = 0$ for all j

Indeed $\quad (b_j | p - \sum_k (b_k|p) b_k) = (b_j|p) - \sum_k (b_j|b_k)(b_k|p)$

$$= (b_j|p) - (b_j|p) = 0$$

Claim b) $\quad (z|\tilde{w}) = 0$ for all $\tilde{w} \in W$

Indeed $\quad (z | \sum t_k b_k) = \sum t_k (z|b_k) = 0$

Claim c) $|p-\tilde{w}|^2 \geq |p-w|^2$ for all $\tilde{w}$
and equality only holds for $w=\tilde{w}$

Indeed,

$$|p-\tilde{w}|^2 = |p-w+(w-\tilde{w})|^2 = (p-w|p-w) + \overbrace{(p-w|w-\tilde{w})}^{=0} + |w-\tilde{w}|^2 + \underbrace{(w-\tilde{w}|p-w)}_{=0}$$

$$\geq |p-w|^2 + |w-\tilde{w}|^2$$

"=" holds exactly when $w=\tilde{w}$!

Thus $w = \sum (b_k|p)\, b_k$ is the solution and

$|p-w|$ is the minimal distance. $\square$

Def $W \subseteq V$ We call

$$Proj_{\bar{W}}(p) = \sum (b_k|p)\, b_k$$

the projection of p onto $\bar{W}$. This projection does not depend on the choice of the orthonormal basis

z perpendicular to $\bar{W}$

Def $$comp_v(p) = \frac{(v|p)\,v}{(v|v)} = Proj_W(p)$$
where $W = \{tv : t\in\mathbb{R}\}$ is a subspace.

Comment : let V be an inner product space $W \subseteq V$ a finite dimensional subspace $p \in V$ be a point.

By choosing ONB (= orthonormal basis) $b_1 \ldots b_k$ for W and the additional orthonormal vector

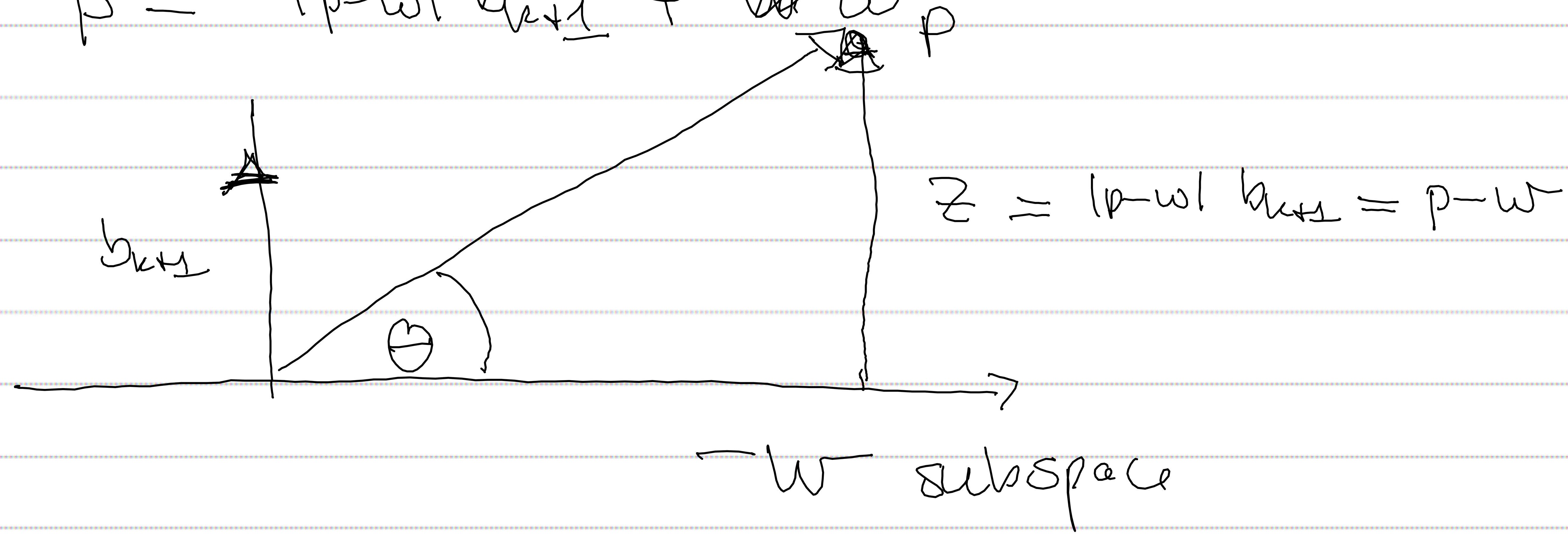
$$b_{k+1} = \frac{p-w}{|p-w|} \qquad w = \sum_k (b_k|p) b_k$$

we may transport the whole picture to $\mathbb{R}^{k+1}$ $\quad \mathbb{R}^k$ corresponds to W

via the map $(t_1, \ldots, t_k) \mapsto \sum_k t_k b_k$

$(0 \ldots 0 1)$ corresponds to b_{k+1}

$$p = |p-w| \, b_{k+1} + \cancel{w} \, w$$



Angles between p and w : $\boxed{\cos \theta_k = \frac{(b_k|p)}{\|p\|}}$