IV Inner product space and distance

→ section 3

$(V, (\ ))$ Inner product space

Definition: The length of a vector is given by

$$|v| = (v, v)^{1/2}$$

Lemma $|(v, w)|^2 \leq |v|^2 |w|^2$

Proof $(v + sw, v + sw) \geq 0$ therefore

$$(v, v) + 2s(v, w) + (w, w)s^2 \geq 0$$

$$(v, v) + s^2(w, w) \geq -2s(v, w) \quad \text{for all } s$$

Choose s such that $-s(v, w) \geq 0$

and $(v, v) = s^2(w, w)$

$$s = \pm \frac{|v|}{|w|}$$

Then $2|v|^2 \geq 2|(v, w)| \frac{|v|}{|w| + \varepsilon}$

Then $|(v,w)| \le |v|(|w|+\varepsilon)$. let $\varepsilon \to 0$ $\square$

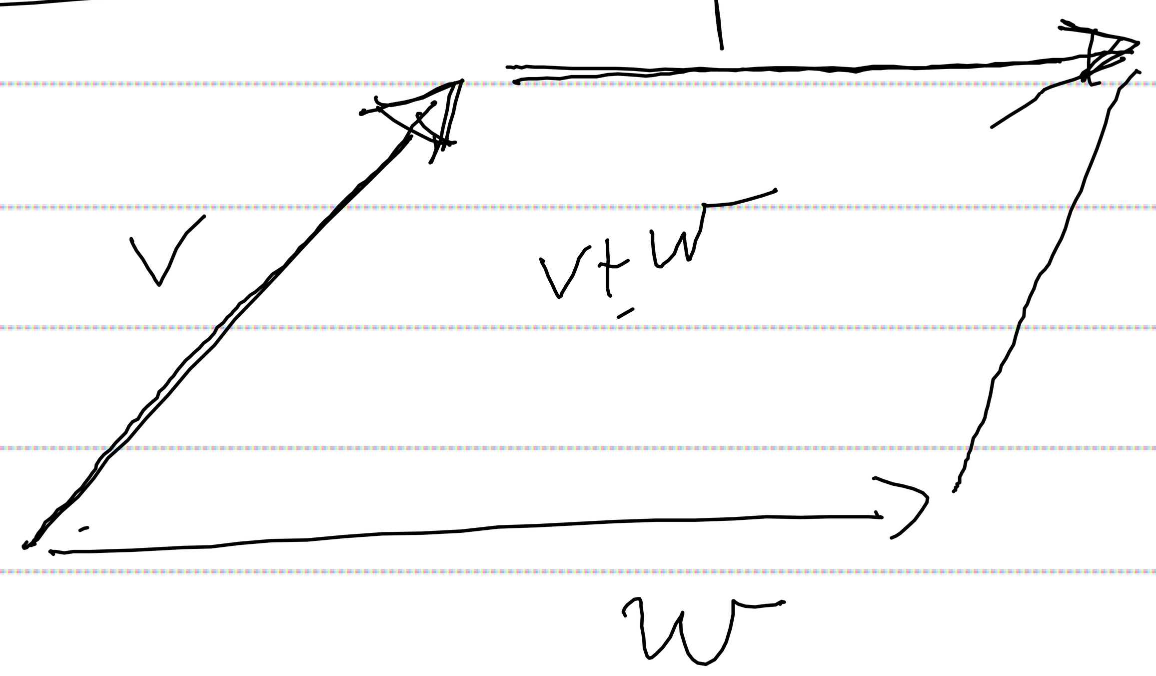
<u>Lemma</u> $|v+w| \le |v| + |w|$

Proof $(v+w, v+w) = |v|^2 + 2(v,w) + |w|^2$

$$\le |v|^2 + 2|v||w| + |w|^2$$

$$= (|v| + |w|)^2$$

<u>Geometric interpretation</u>



length of diagonal smaller than sides

$|(x,y)| = \sqrt{x^2+y^2}$ euclidean length

HW $|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$

Why Parallelogram equality?

Distance to subspace

Problem $W \subseteq V$ subspace

that means

$w_1, w_2 \in W$, $\lambda \in \mathbb{R}$ then $w_1 + \lambda w_2 \in W$

$x \in V$ $x \notin W$

Find $w \in W$ such that

$|x - w|$ is minimal

Fact 1) $w_0 \in W$ with minimal distance

Then $(x - w_0, w) = 0$ for all $w \in W$

Proof $|x - w_0|^2 \leq |x - w|^2 = (x - w, x - w)$

Also $\underbrace{|x - (w_0 + s w)|^2}_{f(s)} \geq |(x - w_0)|^2$

$\boxed{f'(0) = 0}$

$$f(s) = (x-(w_0+sw),\ x-(w_0+sw))$$

$$= (x-w_0,\ x-w_0) + s(x-x_0, w) + s(w, x-x_0) + s^2(w,w)$$

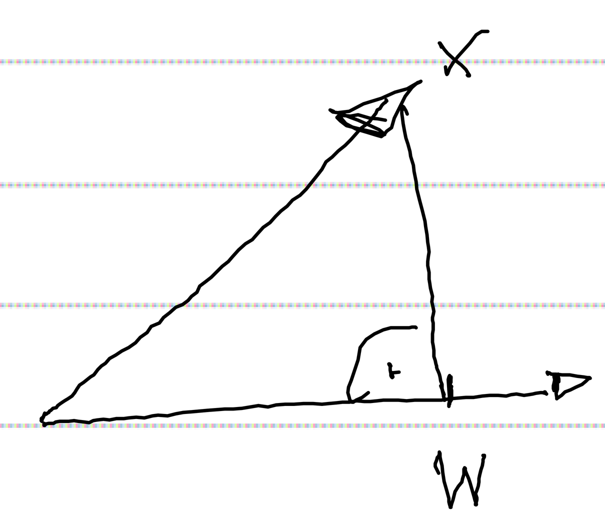
$$f'(s) = 2(x-x_0, w) + 2s(w,w)$$

$$f'(0) = 2(x-w_0, w) \qquad \square$$

<u>Fact 2</u> There is a w_0 such that $(x-w_0, w) = 0$ for all w

<u>Proof</u> Let $b_0, \ldots, b_k$ a basis for W

(here we assume finite dimension)



Define $w_0 = \cancel{\quad} \sum_k (b_k, x) b_k$

$$(\cancel{x\,w_0}, b_j) = \left(\sum_k (b_k, x) b_k, b_j\right) = (b_j, x) = (x, b_j)$$

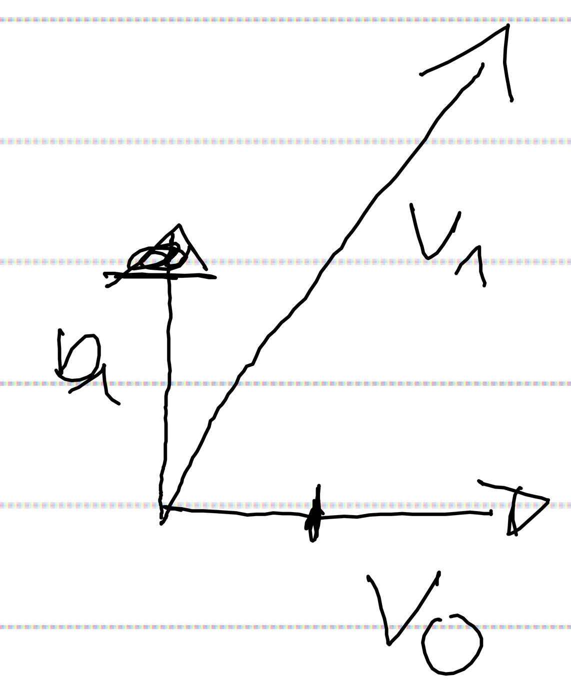
$(x - w_0, b_j) = 0$ for all j

$$\left(x - w_0, \sum t_j b_j\right) = 0$$

Therefore w_0 is the best approximation point. $\square$

Recall Gram-Schmidt method

$v_0 \dots v_n$



next vector orthogonal to that

$$b_0 = \frac{v_0}{|v_0|}$$

$$b_1 = \frac{w_1}{|w_1|} \qquad w_1 = v_1 - (b_0 v_1) b_0$$

and so on

Example $\quad W = \{(1,1,0), (1\ 1\ 1)\}$

$$= \{ s(1,1,0) + t(1,1,1) : t, s \in \mathbb{R} \}$$

$X = (1,-1,1)$ best approximation

1 Step: Find an ONB

$$b_0 = \left(\tfrac{1}{\sqrt2}, \tfrac{1}{\sqrt2}, 0\right)$$

$$w_1 = (1,1,\cancel{0}) - \left((1,1,0), \left(\tfrac{1}{\sqrt2}\ \tfrac{1}{\sqrt2}\ 0\right)\right)\left(\tfrac{1}{\sqrt2}\ \tfrac{1}{\sqrt2}\ 0\right)$$

$$= (1,1,1) - \sqrt2 \left(\tfrac{1}{\sqrt2}\ \tfrac{1}{\sqrt2}\ 0\right)$$

$$= (1\ 1\ \cancel{0}) - (1\ 1\ 0) = (0,0,1) = b_1$$

Step2 <u>Best</u> <u>approximation</u>

$$W = ((1,-1,1), b_0)\, b_0 + ((1,-1,1), b_1)\, b_1$$

$$= b_1 = (0,0,1)$$

Example $V = \{ f : f^{(3)} = 0 \}$ $\quad (\ , \ ) = \int_{-1/2}^{1/2} f(x) g(x)\, dx$

$$\operatorname{dist}(x^2, \{1, x\}) = \inf | x^2 - (\alpha 1 + \beta x) |$$

$\uparrow$ functions

$$= \inf_{\alpha,\beta} \left(\int_{-1/2}^{1/2} | x^2 - (\alpha 1 + \beta x) |^2 \, dx \right)^{1/2}$$

$$= \inf_{\alpha,\beta} \left(\int_{-1/2}^{1/2} (x^2 - \alpha \mp \beta x)^2 \, d\alpha \right)^{1/2}$$

$$\| \quad \left(\int (x^4 + \alpha^2 + \beta^2 x^2 - 2x^2\alpha - 2\beta x^3 + \beta^2 x^2)\, d\alpha \right)^{1/2}$$

$$= \inf_{\alpha,\beta} \left(2\left(\tfrac{1}{2}\right)^5 + \alpha^2 + \beta^2 2\left(\tfrac{1}{2}\right)^3 - 4\alpha\left(\tfrac{1}{2}\right)^3 - 4\beta\left(\tfrac{1}{2}\right)^4 + \beta^2 2\left(\tfrac{1}{2}\right)^3 \right)^{1/2}$$

$$= \inf_{\alpha,\beta} \left(\frac{1}{16} + \alpha^2 + \frac{\beta^2}{4} - \frac{\cancel{4}\alpha}{2} \cancel{+} \frac{\beta}{4} + \frac{\beta^2}{4} \right)^{1/2}$$

$$= \inf_{\alpha,\beta} \left(\frac{1}{16} + \alpha^2 + \frac{\beta^2}{2} - \frac{\alpha}{2} - \frac{\beta}{4} \right)^{1/2}$$

Our method $\quad V_0 = 1 \quad v_1 = x$

$b_0 = \underline{1} \qquad \int_{-1/2}^{1/2} x \, dx = 0$

$b_1 = \frac{W_1}{|W_1|} \qquad w_1 = V_1 - (v_1, b_0) b_0 = \underline{V_1}$

$$\left(\int_{-1/2}^{1/2} x^2 \, dx\right)^{1/2} = \left(\frac{2}{3}\left(\frac{1}{2}\right)^3\right)^{1/2} = \frac{1}{2\sqrt{3}}$$

$= 2 \underline{v_1}$

$$W_{app} = (x^2, b_0) b_0 + (x^2, b_1) b_1$$

$$= \left(\int_{-1/2}^{1/2} x^2\right) \underline{1} + \left(12 \int_{-1/2}^{1/2} x^3 \, dx\right) x$$

$$= \frac{1}{4} + \left(4 \; 2 \, \frac{x^4}{4}\Big|_0^{1/2}\right) x$$

$$\boxed{= \frac{1}{4} + \frac{3}{8} x}$$

$$12 \int_{-1/2}^{1/2} x^3 \, dx = 24 \int_0^{1/2} x^3 \, dx = \frac{24}{4} \, x^4 \Big|_0^{1/2}$$

$$= 6 \, \frac{1}{2^4} = \frac{6}{16} = \frac{3}{8}$$

V Angles and area

Fact v_0, v_1 two vectors in innerproduct space
Then

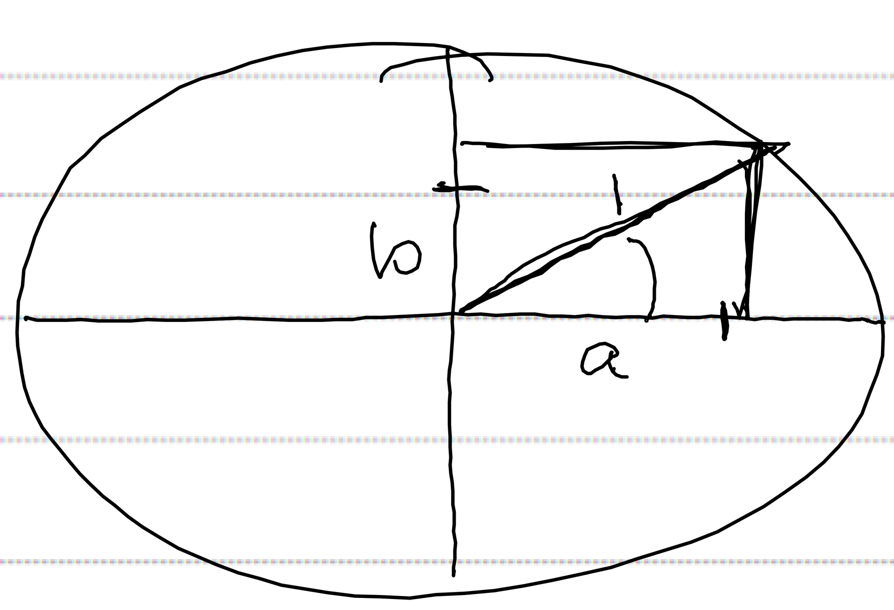
$$(v_0, v_1) = |v_0|\,|v_1| \cos\theta$$

Proof We may assume $v_0 v_1 \subseteq \mathbb{R}^2$
and $v_0 = (t, 0)$ $t > 0$ $t = |v_0|$
$v_1 = (a, b)$

$$(v_0, v_1) = ta = |v_0|\, a$$

$$= |v_0| \frac{a}{\sqrt{a^2+b^2}} \sqrt{a^2+b^2}$$

$$= |v_0|\,|v_1| \frac{a}{\sqrt{a^2+b^2}}$$



$a = \cos\theta$

essentially how we may define cosine geometrically
($\longrightarrow$ complex numbers)

Question 2 Why may we assume that $V = \mathbb{R}^2$

Because we can always apply Gram-Schmidt and find

$$T : \mathbb{R}^2 \longrightarrow V$$

$$T(x,y) = x b_0 + y b_1$$

and this linear map preserves inner product. !

$$(T(x,y), T(x',y')) = xx' + yy'$$

$$\parallel$$

$$(x b_0 + y b_1, x' b_0 + y' b_1)$$

$$\parallel$$

$$xx'(b_0,b_0) + xy'(b_0,b_1) + yx'(b_1,b_0) + yy'(b_1,b_1)$$

$$\parallel$$

$$xx' + yy'$$