Calculus

Part 1 Review — and continuity for vector-valued functions.

Let f a function with values in $\mathbb{R}$

$$\mathrm{dom}(f) = \{x \in \mathbb{R} : f(x) \text{ is defined}\}$$

Remark In order to be continuous at a point $p \in \mathbb{R}$ we shall assume that for some $a > 0$

$$|p - x| < a \implies x \in \mathrm{dom}(f)$$

(That means the dom(f) is open: with the every point $p \in \mathrm{dom}(f)$ there is a little interval $(p-a, p+a)$ also contained in dom(f))

Definition f is continuous at p if

$$x_n \xrightarrow[\text{converges to}]{} p \implies f(x_n) \xrightarrow[\text{converges to}]{} f(p)$$

Sometimes this is expressed in $\varepsilon - \delta$ form

Given $\varepsilon > 0$ there exists $\delta > 0$ so that

$$|x - p| < \delta \quad \underset{\text{(forces)}}{\text{implies}} \quad |f(x) - f(p)| < \varepsilon$$

- Continuity ~~with~~ for functions in inner product Spaces V

$f: \mathrm{dom}(f) \subseteq \mathbb{R} \longrightarrow V$

$p \in \mathrm{dom}(f)$ f is called continuous at p

if given ~~every~~ $\varepsilon > 0$ there exists $\delta > 0$ such that

$|x-p| < \delta$ implies $|f(x) - f(p)| < \varepsilon$

alternatively

$\underbrace{x_n \longrightarrow p}_{\text{converge}} \implies f(x_n) \xrightarrow{\text{in } V} f(p)$

What means $V_n \longrightarrow V$ (read V_n converges to v)

Answer $\boxed{|V_n - V| \longrightarrow 0 \text{ real numbers.}}$

- How can we check this? (over?)

Ex $f: \mathbb{R} \longrightarrow \mathbb{R}^2$ $f(x) = (\cos x, e^{-x})$

(not linear map)

Note $V_m = (V_1^m, \ldots, V_n^m)$ a vector in $\mathbb{R}^n$ (n-space)

then V_m converges to $(V_1, \ldots, V_n) = V$

if and only if

$$|V_j^m - V_j| \longrightarrow 0 \quad \text{for all } j$$

if and only if

V_j^m converges to V_j for all $j = 1 \ldots n$

"coordinate wise convergence"

Application: $f : \text{dom}(f) \longrightarrow \mathbb{R}^n$ is given by n functions $f_j : \text{dom}(f) \longrightarrow \mathbb{R}$

$$f(x) = (f_1(x), \ldots, f_n(x))$$

f is continuous at p if all the functions $f_1, \ldots, f_n$ are continuous at p. ! (Thats easy)

Thus $f(x) = (\cos x, e^{-x})$ is continuous at every point.

Part 2 Review Differentiability and Taylor

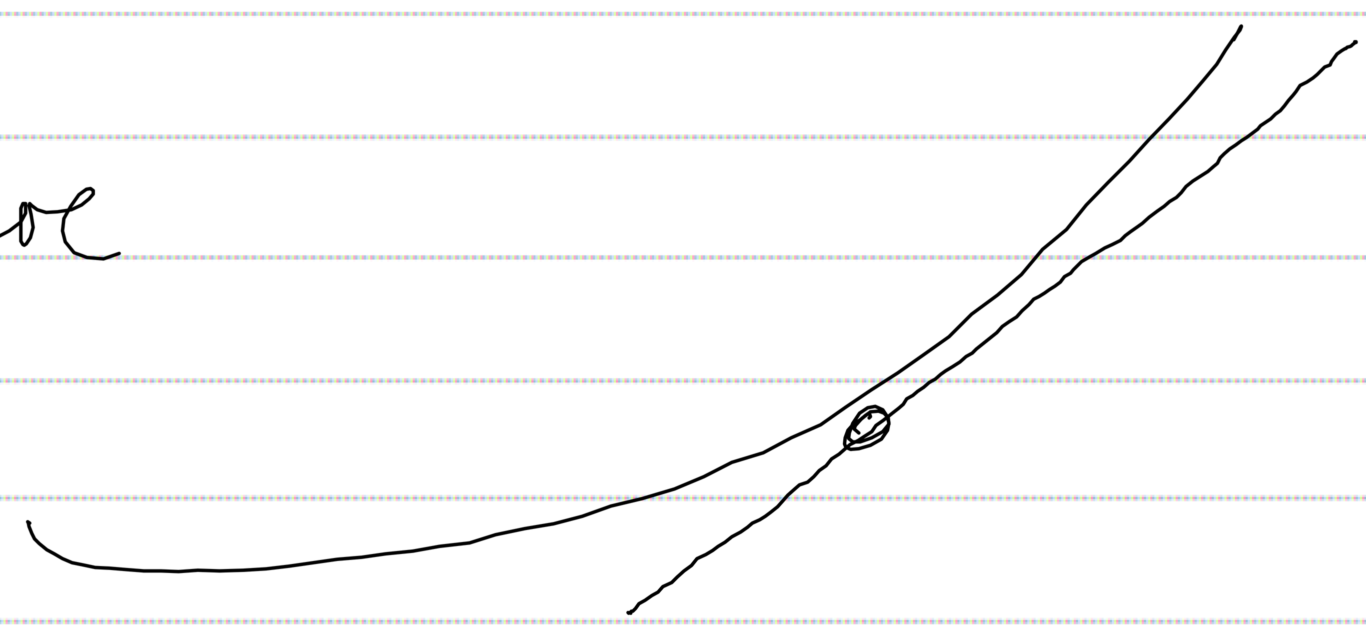
$f: \mathrm{dom}(f) \longrightarrow \mathbb{R}$

f is called <u>differentiable at</u> $p \in \mathrm{dom}(f)$

if there exist slope s such that

$$\frac{|f(x)-f(x_0)-s(x-x_0)|}{|x-x_0|} \xrightarrow[x \to x_0]{} 0$$

picture



$\boxed{\text{line equation: } y = f(x_0)+s(x-x_0)}$

line is close to function

read as

$$|\underbrace{f(x)}_{\text{function}} - \underbrace{[f(x_0)+s(x-x_0)]}_{\text{line}}| < \varepsilon |x-x_0|$$

for $|x-x_0| < \delta$

for every $\varepsilon > 0$ I can find δ such that

Taylor formula

$$\left|\frac{f(x) - \left[f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\right]}{|x-x_0|^n}\right| \xrightarrow[x \to x_0]{} 0$$

means that we can approximate a function f at $x_0 (=p)$ by a polynomial of order n and the error is smaller than $|x-x_0|^n$

Recall there was something like

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0) = \frac{1}{n!}\int_{x_0} f^{(n+1)}(x)\,(x-x_0)^n \, dx$$

$$\boxed{\text{Hence } (n+1) \text{ continuous differentiable} \implies \text{Taylor formula of order } n.}$$

(maybe better)

Part 3 Differentiation for vector-valued function defined on $\mathbb{R}$

Definition $f : \mathrm{dom}(f) \overset{\subseteq \mathbb{R}}{\longrightarrow} V$ (inner product space)

is called differentiable at p if there exists

a vector $v \in V$ such that

$$f(x) = f(x_0) + v(x-x_0) + \mathrm{error}(x-x_0)$$

so that

$$\boxed{\lim_{x \to x_0} \frac{|\mathrm{error}(x-x_0)|}{|x-x_0|} = 0}$$

or

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0) - v(x-x_0)|}{|x-x_0|} = 0$$

Test: $f : \mathrm{dom} f \longrightarrow V \cong \mathbb{R}^n$ is given by

n functions $f(x) = (f_1(x), \dots, f_n(x))$

(called coordinate functions

If all the f_j's are differentiable at p, then so is f

$$f'(p) = (f_1'(p), \dots, f_n'(p)) \in V$$

Rem The derivative is a vector !

Taylor formula: Let $f = (f_1, \dots, f_n)$ be an $\mathbb{R}^n$ valued function such that all the functions $f_1, \dots, f_n$ are $(m+1)$ times continuously differentiable. Then

$$\frac{\left| f(x) - f(x_0) - \sum_{k=1}^{m} \frac{f^k(x_0)}{k!}(x-x_0)^k \right|}{|x-x_0|^m} \longrightarrow 0 \quad x \to 0$$

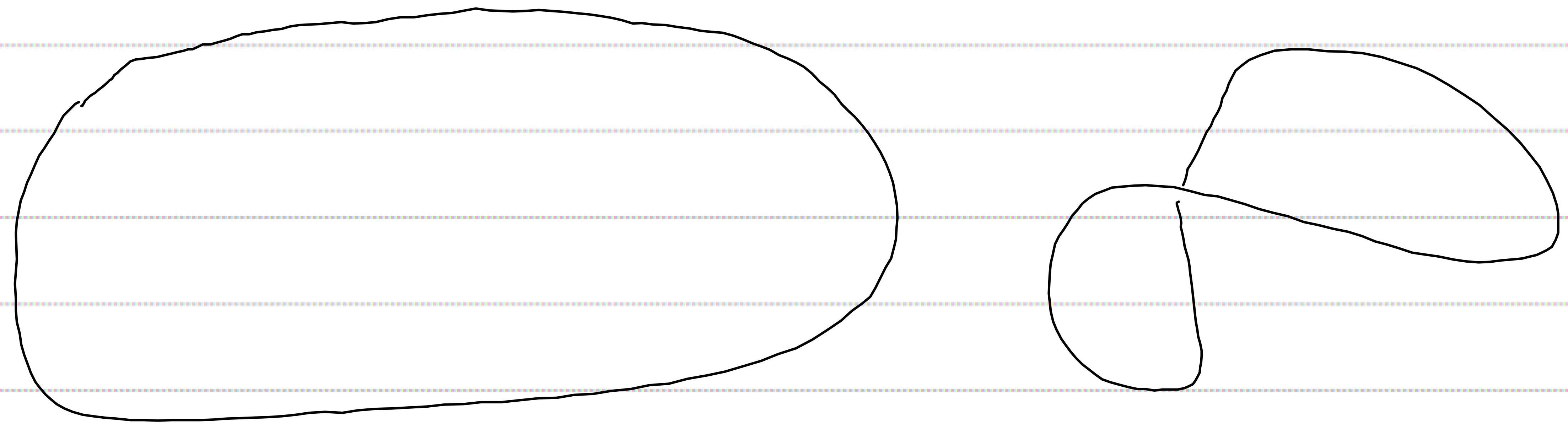
or

$$\lim_{x \to x_0} \frac{\left| f(x) - \cancel{f(x_0)} \sum_{k=0}^{m} \frac{f^k(x_0)}{k!}(x-x_0)^k \right|}{|x-x_0|^m} = 0$$

"copy cat"

that needs improvement.

Application – Curves



Defenition a curve is the image $f([a,b])$ of a function

$f: \mathbb{R} \longrightarrow \mathbb{R}^n$ by an interval

Attention: Several functions lead to same image

$$f(\theta) = (\cos\theta, \sin\theta)$$

$$f[-\pi, \pi] = S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$$

$$F(\theta) = (\cos 3\theta, \sin 3\theta)$$

$$F[-\pi/3, \pi/3] = S^1.$$

Remark let $C = f[a,b]$ $\quad x_0 \in (a,b)$

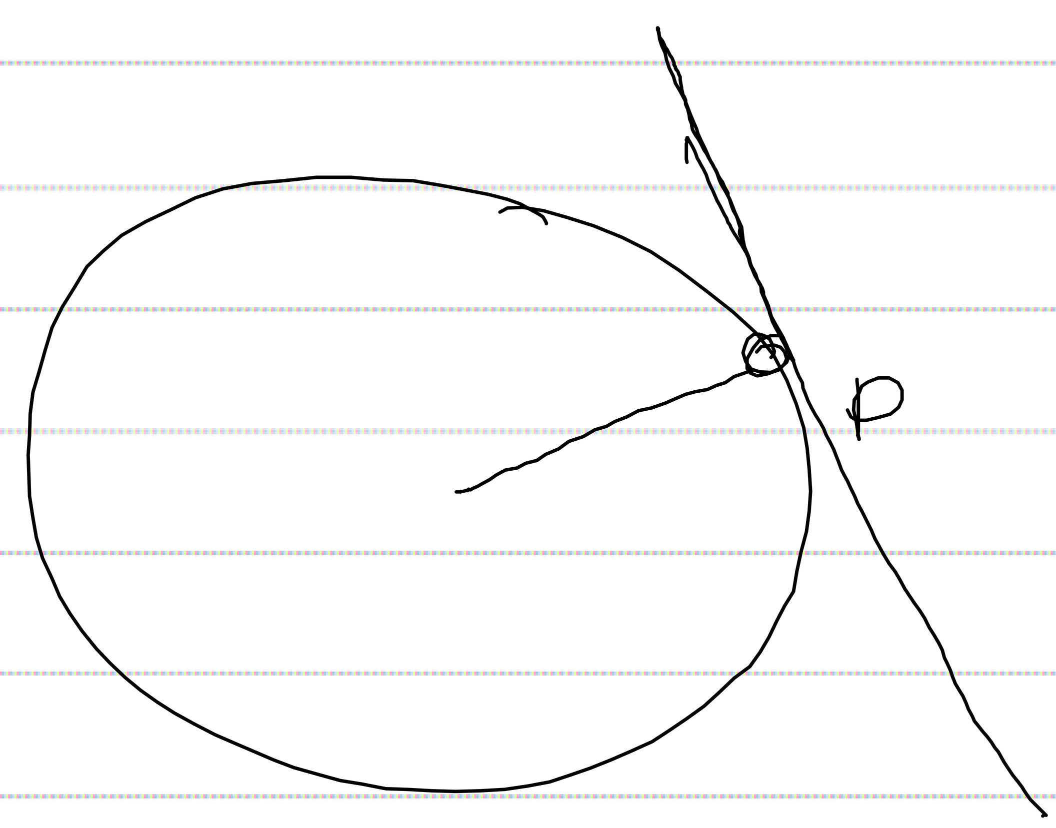
$f'(x_0) \neq 0$ Then $f'(x_0)$ is the direction

of the tangent line of C at $f(x_0)$

Example $f(\theta) = (\cos\theta, \sin\theta)$

$p = f(\theta)$

$f'(\theta) = (\sin\theta, -\cos\theta)$



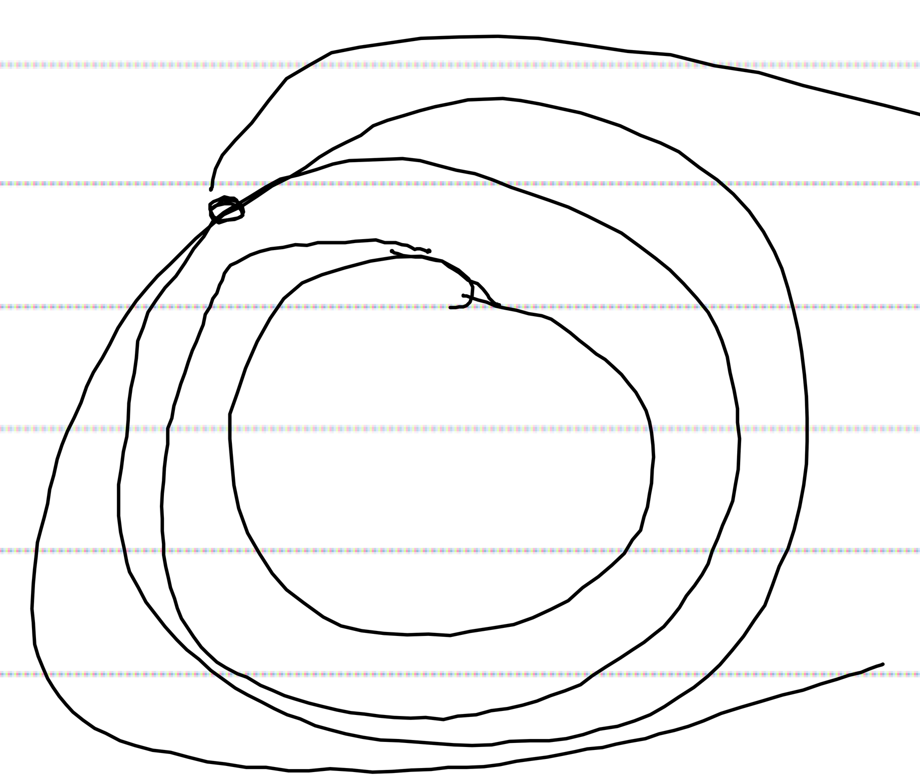
In general

$\ell = \{ f(x_0) + t f'(x_0) \; : \; t \in \mathbb{R} \}$ is the line and

$$| f(x) - [f(x_0) + f'(x_0)(x - x_0)] | < \varepsilon |x - x_0|$$

for $|x - x_0| < \delta$ means that line is close to curve when $f(x)$ is close to $f(x_0)$.

Attention



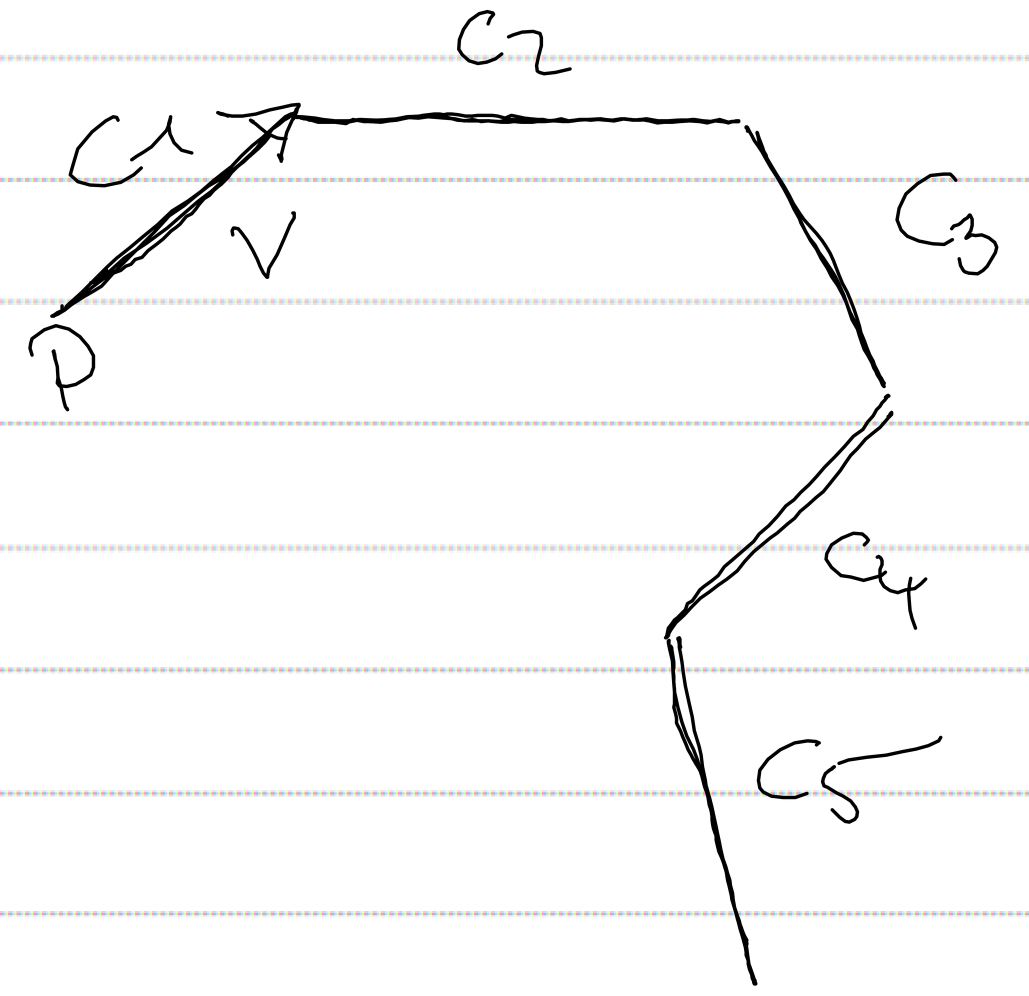
be careful

with double points

Note only direction, not length is determined by curve.

Length of a curve

C is the union of simple curves



C_1 : $f(x) = p + tv$ $\quad 0 \le t \le 1$

$$|C_1| = |v| = \int_0^1 |f'(t)| \, dt$$

In general

$$L(C) = \int_a^b |f'(t)| \, dt$$

whenever $C = f[a,b] = \{f(t) : a \le t \le b\}$ and the curve is only run through once

(or $|f'(t)| > 0$ and f is injective)

Ex $|S^1| = \int_0^{2\pi} |f'(\theta)|\, d\theta = 2\pi$

$f(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $f'(\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

$|f'(\theta)| = 1$

More examples $\longrightarrow$ Homework.

Dimension 2 Frame bundle

In dimension 2 we have uniform method to determine an orthogonal vector to a given vector

$v = \begin{pmatrix} a \\ b \end{pmatrix}$ $v^\perp = \begin{pmatrix} -b \\ a \end{pmatrix}$ read perp

Normalized

$v = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $v^\perp = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

Conclusion: let c be curve so that

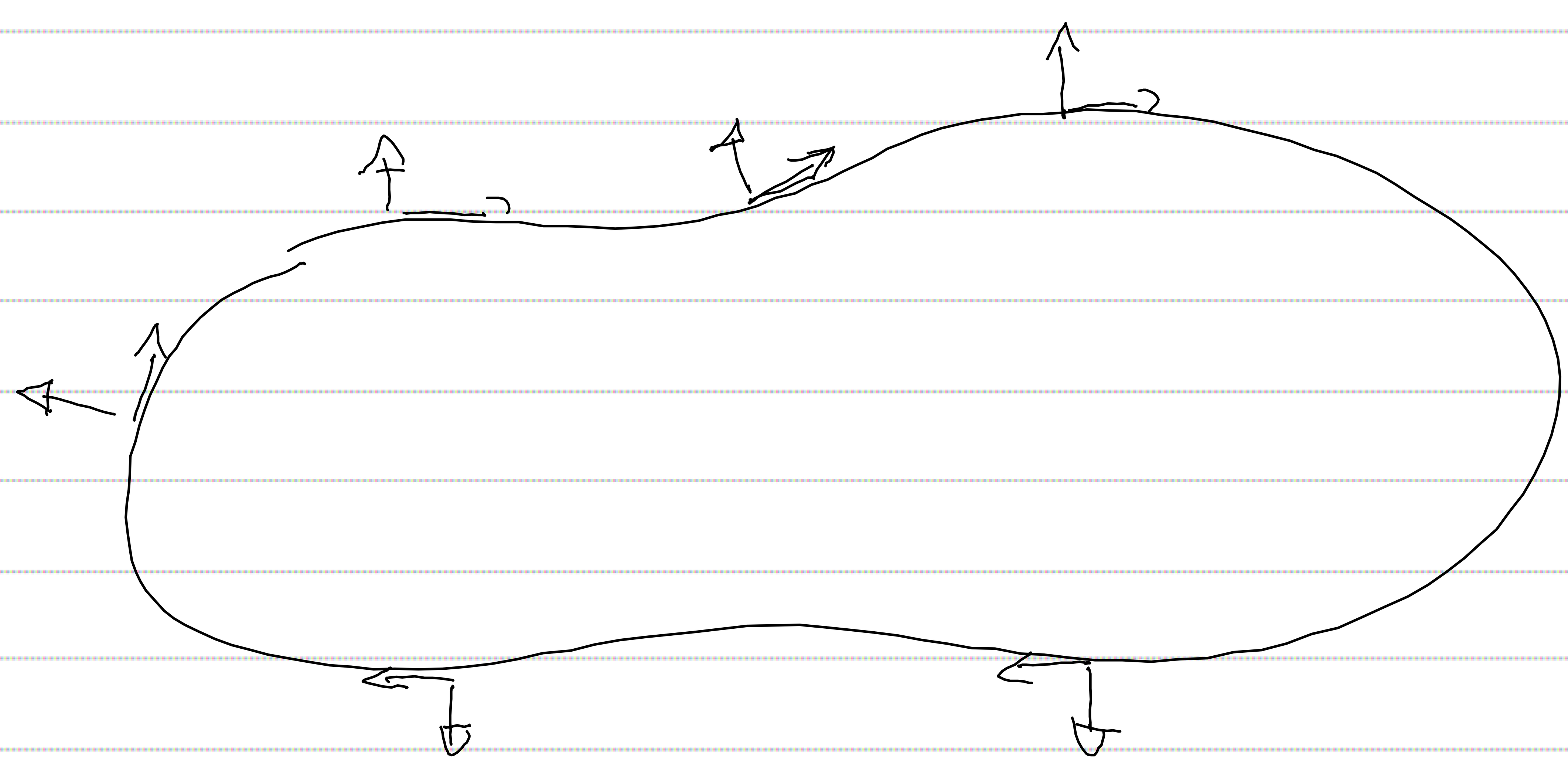
$c = f[a,b]$ f injective $f' \neq 0$

Then every point of c carries a tangent and a normal vector "continuously"

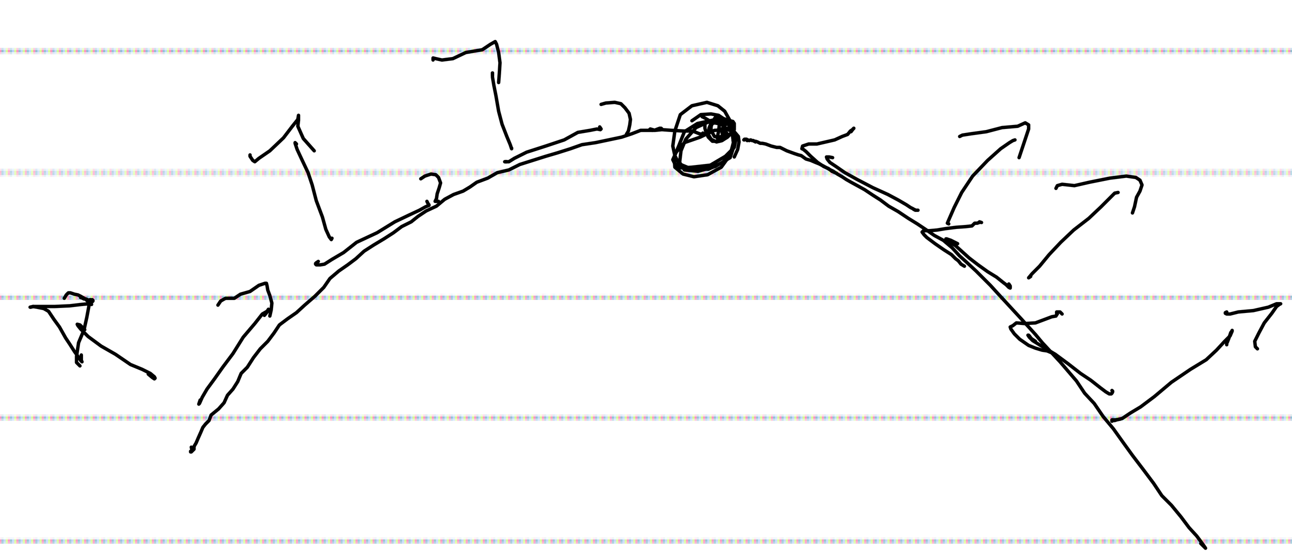
Indeed

$$p = f(t) \qquad t(p) = \frac{f'(t)}{|f'(t)|} \qquad n(p) = \left(\frac{f'(t)}{|f'(t)|}\right)^{\perp}$$

determines such a choice.



Since f is continuous and $f' \neq 0$ we have to choose an orientation.



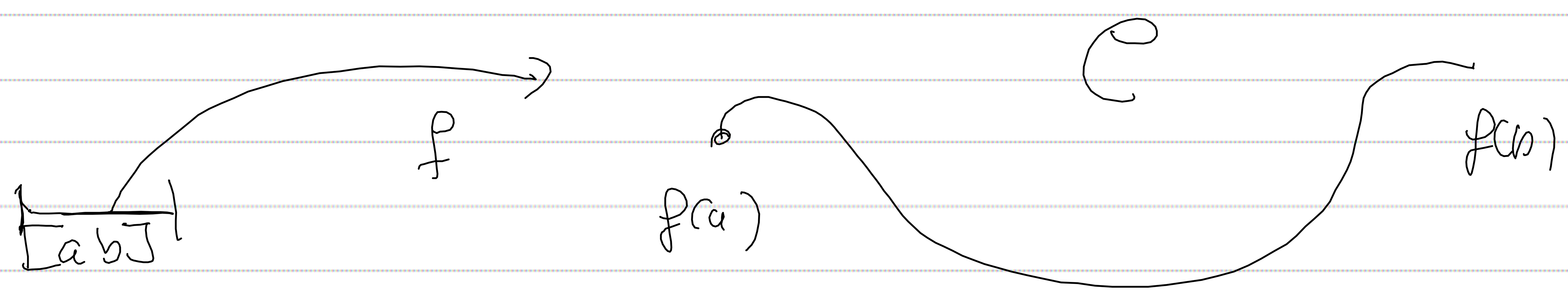
is excluded

Curves

Definition A curve $\mathcal{C}$ is given by a continuous, injective, piecewise differentiable map $f: [a,b] \longrightarrow \mathbb{R}^d$ and

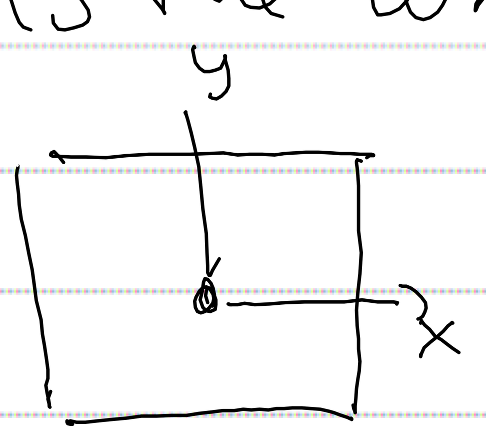
$$\mathcal{C} = \{f(x) : a \le x \le b\} = f[a,b]$$

Note ($f(a) = f(b)$ is allowed (exception to injective) and then $\mathcal{C}$ is called close curve



Ex 1 $f(x) = \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix}$ $f[0, 2\pi] = \mathcal{C} = S^1 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a^2 + b^2 = 1 \right\}$

is the unit circle

Ex 2 

$$f(x) = \begin{cases} \begin{pmatrix} x-1 \\ -1 \end{pmatrix} & 0 \le x \le 2 \\ \begin{pmatrix} 1 \\ x-3 \end{pmatrix} & 2 \le x \le 4 \\ \begin{pmatrix} 5-x \\ 1 \end{pmatrix} & 4 \le x \le 6 \\ \begin{pmatrix} -1 \\ 7-x \end{pmatrix} & 6 \le x \le 8 \end{cases}$$

Ex 3 $F(x) = \begin{pmatrix} \cos(x^2) \\ \sin(x^2) \end{pmatrix}$ $F[0, \sqrt{2\pi}] = f[0, 2\pi]$

Definition $f[a,b] \longrightarrow \mathbb{R}^d$

$$L(C) = \int_a^b |f'(x)|\, dx$$

More generally, let $h: \mathbb{R}^d \longrightarrow \mathbb{R}^m$ continuous

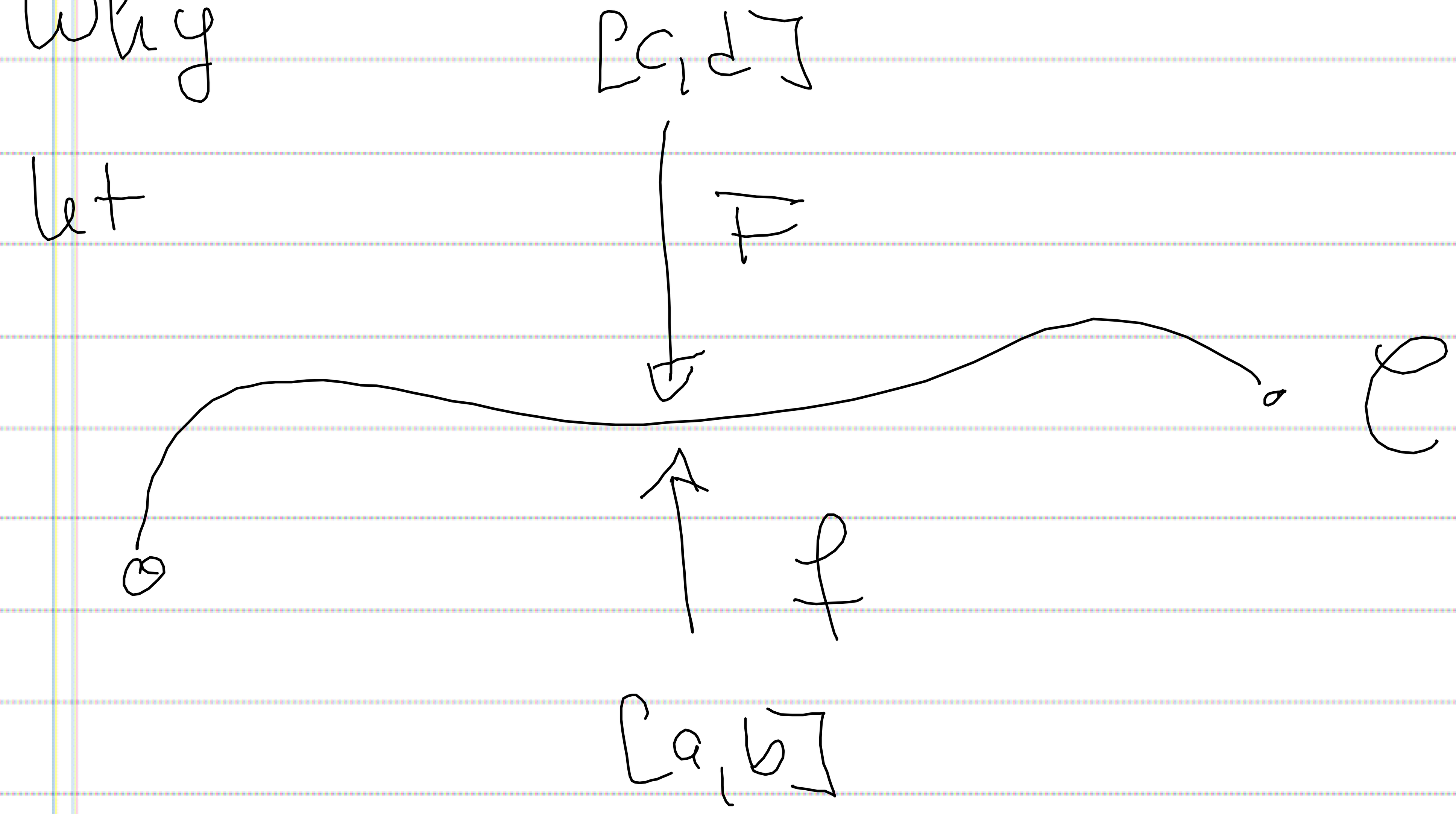
Then we define

$$\int_C h(p)\, d\sigma(p) = \int_a^b h(f(x))\, |f'(x)|\, dx$$

Note This average over vectors along the curve does not depend on the "parametization" f.

Why

let



$f: [a,b] \longrightarrow C \longleftarrow [c,d] : F$ two maps as above

such that $C = f[a,b] = F[c,d]$

Since f is injective the function $f^{-1}: \mathcal{C} \to [a,b]$ is well defined and we find $\gamma: [c,d] \to [a,b]$ such that

$$f(\gamma(s)) = F(s)$$

By chain rule

$$\frac{dF}{ds} = \frac{d\, f \circ \gamma}{ds} = \frac{df}{dx}\,\frac{d\gamma}{ds}$$

Hence

$$\int_c^d |F'(s)|\, h(F(s))\, ds \qquad\qquad F(s) = p = f(x)$$

$$x = \gamma(s)$$

$$= \int_c^d |f'(\gamma(s))|\, h(f(\gamma(s)))\, \left|\frac{d\gamma}{ds}\right| ds$$

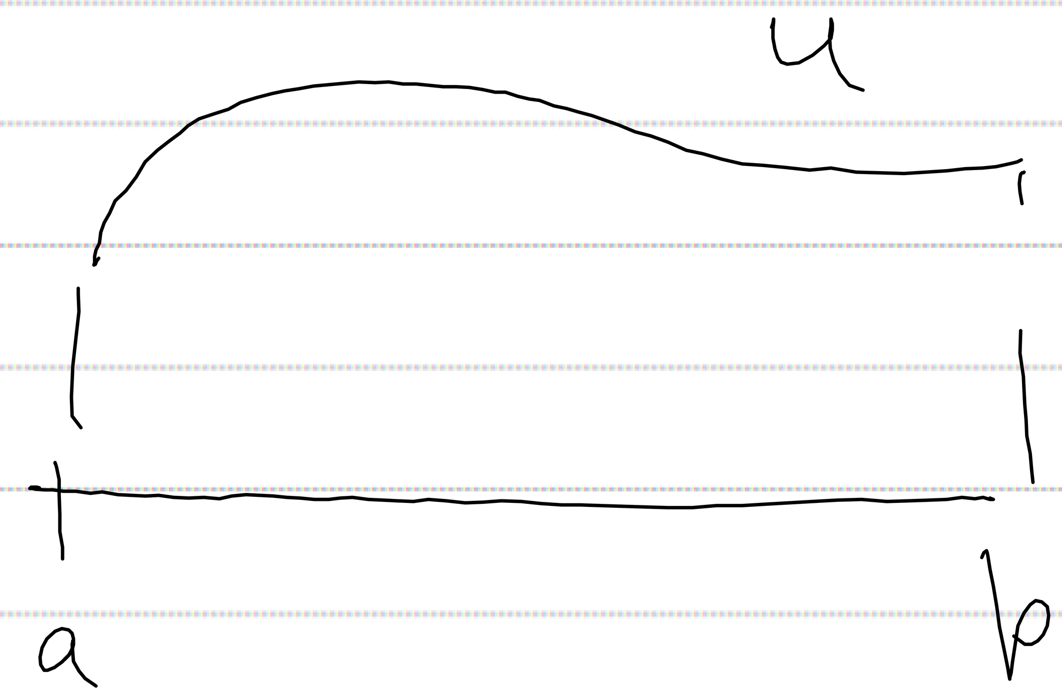
$$= \int_a^b |f'(x)|\, h(f(x))\, dx$$

change of variable

In particular

$$\boxed{L(\mathcal{C}) = \int_a^b |f'(x)|\, dx = \int_c^d |F'(s)|\, ds}$$

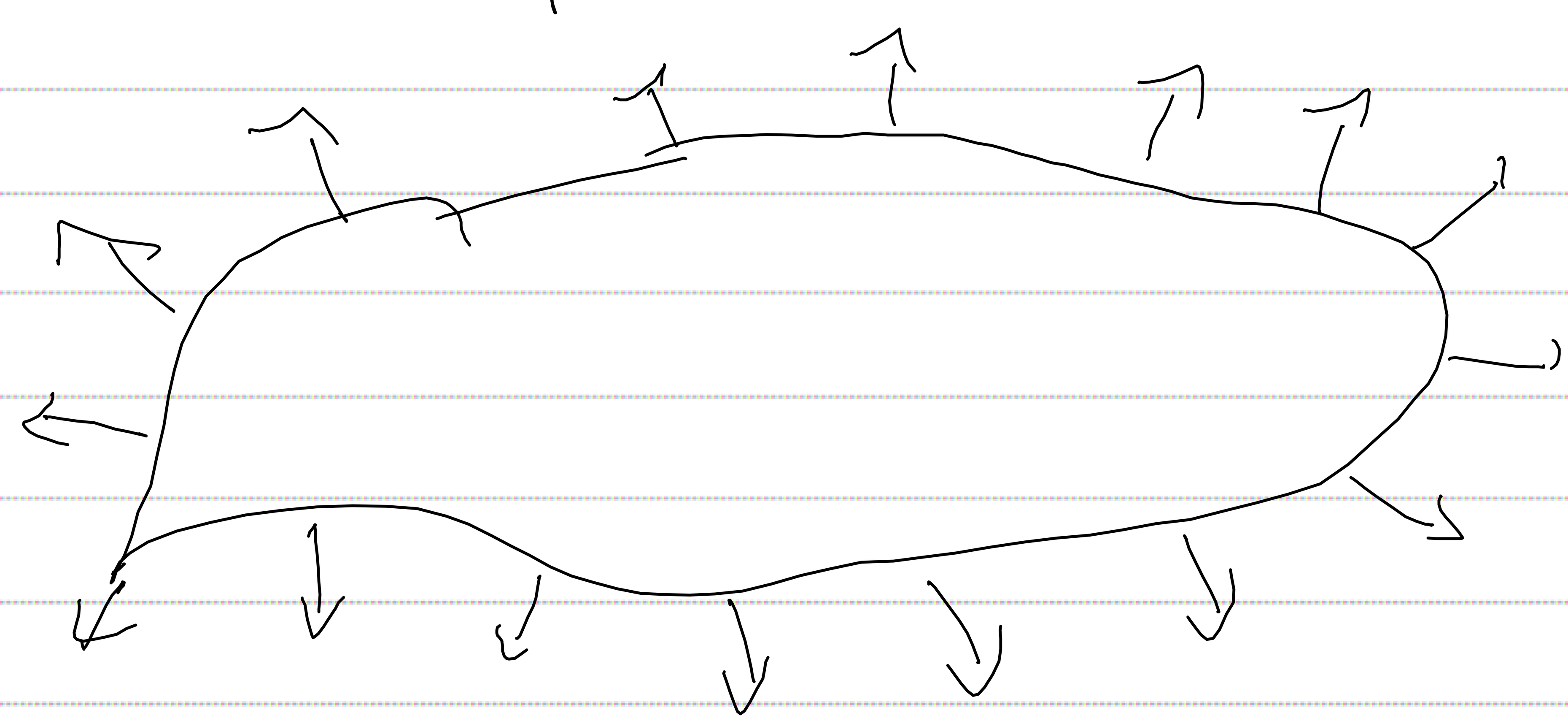
Ex $f(x) = \begin{pmatrix} x \\ u(x) \end{pmatrix}$ $\quad f'(x) = \begin{pmatrix} 1 \\ u'(x) \end{pmatrix}$



$$L(u) = \int_a^b \sqrt{1 + u'(x)^2}\, dx$$

Goal: let C be a close curve in $\mathbb{R}^2$

and $\nu(p)$ the normalized normal vector



Theorem $\boxed{\int_C \nu(p)\, d\sigma(p) = 0}$

Proof or "reason" later

More geometry of curves

Observation: Let $C = f[a,b]$ be a curve and $p \in C$. The (normalized) tangent vector

$$T(p) = \frac{f'(x)}{|f'(x)|} \qquad p = f(x)$$

is determined up to sign.

Indeed,

$$p = f(x) \qquad p = F(s)$$

let $\gamma = f^{-1}F \quad [c,d] \longrightarrow [a,b]$

the "reparametrization" function from above

Then

$$\frac{\frac{dF}{ds}}{\left|\frac{dF}{ds}\right|} = \frac{\frac{df}{dx}(\gamma) \cdot \frac{d\gamma}{ds}}{\left|\frac{df}{dx}(\gamma)\,\frac{d\gamma}{ds}\right|} = \frac{\frac{df}{dx}}{\left|\frac{df}{dx}\right|} \quad \frac{r}{|r|}$$

where $r = \frac{d\gamma}{ds}$ $\quad$ r is a real number and hence

$$\frac{r}{|r|} = \pm 1 \quad !$$

Parametrisation by arc length

let $s(x) = \int_a^x |f'(y)|\,dy$ $\qquad s : [a,b] \longrightarrow [0, L(C)]$

and $F(s) = f(x)$ if $s = s(x)$

Theorem i) $|F'(s)| = 1$

ii) $(F'(s) \,|\, F''(s)) = 0$

Lemma $\dfrac{ds}{dx} = |f'(x)|$ $\qquad \dfrac{dx}{ds} = \dfrac{1}{|f'(x)|}$

$$\frac{d^2s}{dx^2} = \frac{(f''(x) \,|\, f'(x))}{|f'(x)|} \qquad \frac{d^2x}{d^2s} = -\frac{(f''(x) \,|\, f'(x))}{|f'(x)|^4}$$

Proof Let $s^{-1} : [0, L(C)] \to [a,b]$ the inverse function. Then

$x = s^{-1}(s(x))$ implies $\qquad t = s(x)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cancel{s'(t) = x}$

$$1 = \frac{d\cancel{s^{-1}}}{d\cancel{s}} \frac{ds}{dx}$$

here

Hence $\dfrac{dx}{ds} = \dfrac{1}{\frac{ds}{dx}} \stackrel{\downarrow}{=} \dfrac{1}{|f'(x)|}$

Moreover

$$0 = \frac{d}{dx}(1) = \frac{d}{dx}\left(\frac{dx}{ds}\cdot\frac{ds}{dx}\right)$$

$$= \frac{d^2x}{ds^2}\left(\frac{ds}{dx}\right)^2 + \frac{dx}{ds}\frac{d^2s}{dx^2} \quad \text{by product rule}$$

Therefore

$$\frac{d^2x}{ds^2} = -\frac{\frac{d^2s}{dx^2}\frac{dx}{ds}}{\left(\frac{ds}{dx}\right)^2} = -\frac{\frac{(g''(x)\,|\,g'(x))}{|g'(x)|}\frac{1}{|g'(x)|^2}}{|g'(x)|^2}$$

$$= -\frac{(g''(x)\,|\,g'(x))}{|g'(x)|^4}$$

What is $\frac{d^2s}{dx^2}$

$$\frac{ds}{dx} = |g'(x)| = \left(\sum_{k=1}^{d} f_k'(x)^2\right)^{1/2}$$

$$\frac{d^2s}{dx^2} = \frac{1}{2}\left(\sum_{k=1}^{d} f_k'(x)^2\right)^{-1/2}\sum_{k=1}^{d} 2 f_k''(x) f_k'(x)$$

$$\boxed{= \frac{(g''(x)\,|\,g'(x))}{|g'(x)|}}$$

$\square$

Proof of Theorem

$$\boxed{\begin{aligned} \frac{dF}{ds} &= \frac{df}{dx}\,\frac{dx}{ds} = \frac{f'(x)}{|f'(x)|} \\ \left|\frac{dF}{ds}\right| &= \frac{|f'(x)|}{|f'(x)|} = 1 \end{aligned}}$$

$$\frac{d^2F}{d^2s} = \frac{d}{ds}\left(\frac{df}{dx}\,\frac{dx}{ds}\right)$$

$$= \frac{d^2f}{d^2x}\left(\frac{dx}{ds}\right)^2 + \frac{df}{dx}\,\frac{d^2x}{ds^2}$$

$$= \frac{f''(x)}{|f'(x)|^2} - f'(x)\,\frac{(f''(x)\,|\,f'(x))}{|f'(x)|^4}$$

$$= \frac{1}{|f'(x)|^2}\left(f''(x) - \frac{(f''(x)\,|\,f'(x))}{|f'(x)|^2}\, f'(x)\right)$$

Recall Gram-Schmidt for $V_1 = f'(x)$ $V_2 = f''(x)$

Then

$$W_2 = V_2 - \frac{(V_2\,|\,V_1)}{|V_1|^2}\, V_1$$ is orthogonal to V_1

We get

$$\frac{d^2F}{ds} = \frac{w_2}{|g'(x)|^2} = \frac{|w_2|}{|g'(x)|} b_2$$

is a multiple of the second normalized basis vector. $\square$

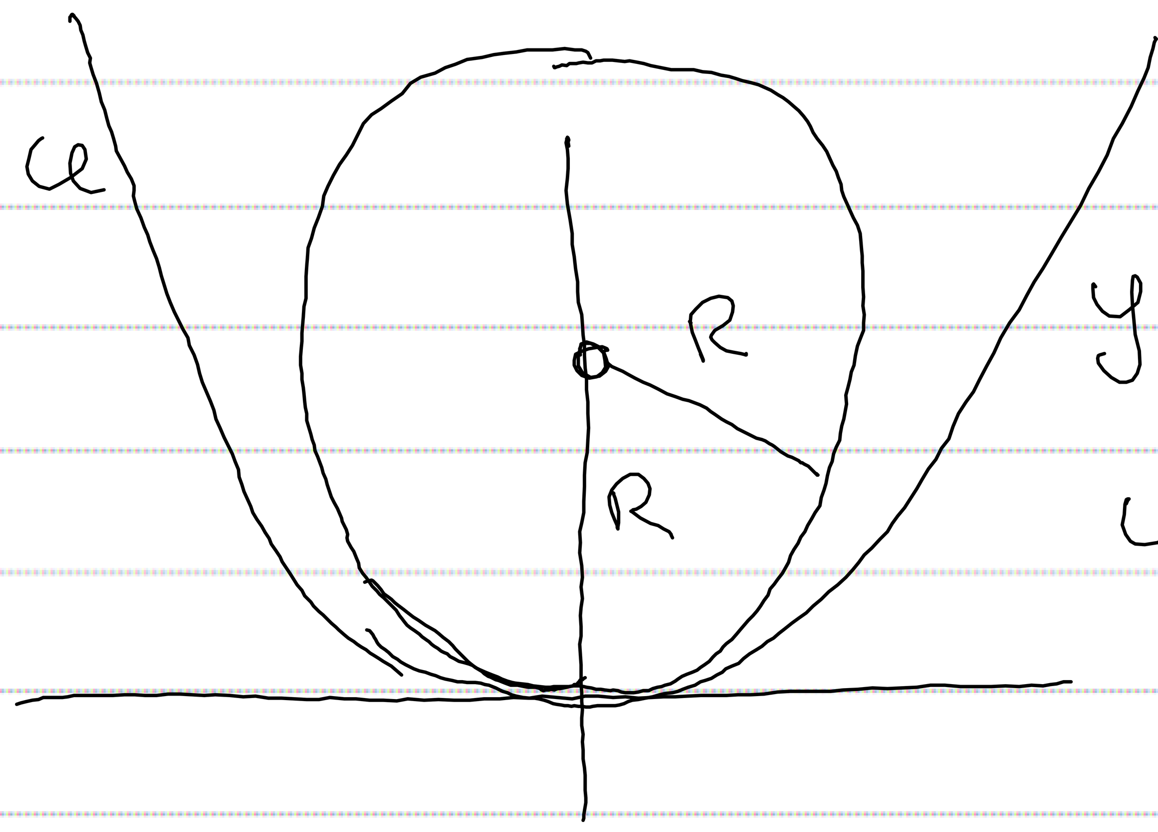
Curvature

in $\mathbb{R}^2$: $h(x) = bx^2$ $h'(0)=0$ $h(0)=0$

Then for $R = \frac{1}{2b}$

$$\left\| \begin{pmatrix} 0 \\ R \end{pmatrix} - \begin{pmatrix} x \\ bx^2 \end{pmatrix} \right\|^2 = x^2 + (R - bx^2)^2$$
$$= x^2 + R^2 - \overbrace{2bRx^2}^{1} + b^2x^4$$
$$= R^2 + b^2x^4 \geq R^2$$

Hence



$y = bx^2$ is outside of disc with radius R

curvature of h with $h(0) = 0 = h'(0)$ is given by $\boxed{|h''(0)|} = \frac{1}{R}$ Inverse of inscribed radius.

For general curves

Just consider $\mathcal{C}$ given by f

(projected to) in the plane of $f'(x)$ and $f''(x)$

Better: use $F'(s)$ and $F''(s)$ $\qquad f(x) = p = F(s)$

obtained from arc length

Now $\quad |F'(s)| = 1$ and hence

$$\kappa(p) = \boxed{|F''(s)|} \qquad (\to |b|)$$

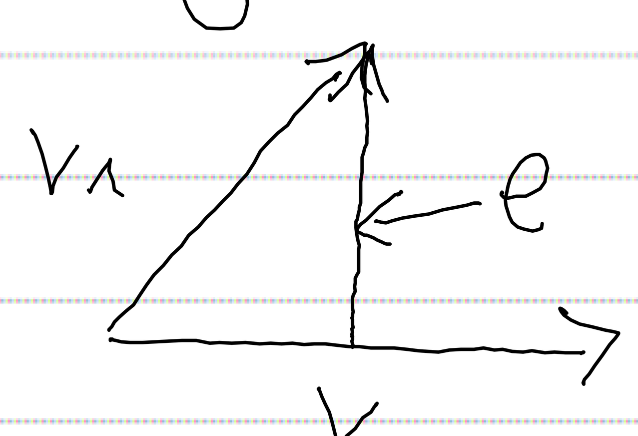
From previous calculation we know that

$$|F''(s)| = \left|\frac{d^2F}{d^2s}\right| = \frac{\left(\left| f''(x) - \frac{(f''(x)|f'(x))}{|f'(x)|^2} f'(x)\right|\right)}{|f'(x)|^2} \; \frac{|f'(x)|}{|f'(x)|}$$

$$\kappa(p) = \boxed{\frac{|f'(x) \times f''(x)|}{|f'(x)|^3}}$$

Independent of parametrization

Why? $\quad$ Consider



Recall $e = \left| v_2 - \frac{(v_1|v_2)}{(v_1|v_1)} v_1 \right|$

$|v_1 \times v_2| = \text{area} = e\,|v_1|$

better $\boxed{\kappa(p) = \dfrac{\left| (f'(x)\,|\,f'(x))\, f''(x) - (f''(x)\,|\,f'(x))\, f'(x) \right|}{(f'(x)\,|\,f'(x))^2}}$

In general

$$\text{area}(v_1, v_0) = \left| v_2 - \frac{(v_1 | v_2) v_1}{(v_1 | v_1)} \right| |v_1|$$

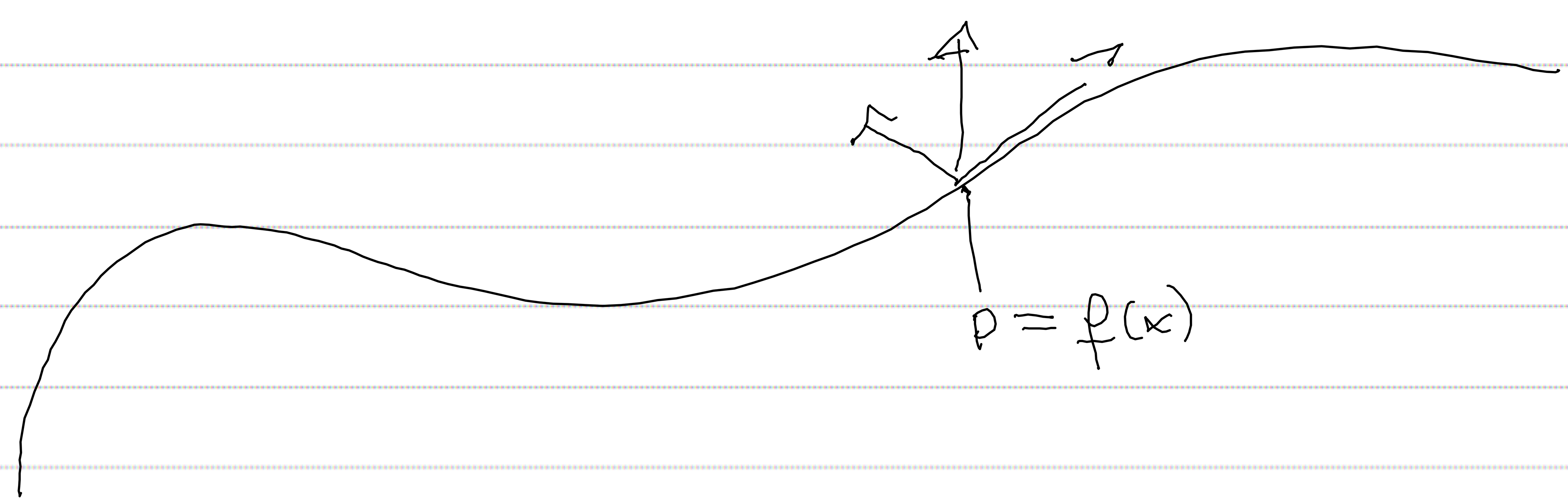
this formula hold not only in $\mathbb{R}^3$ but in every inner product space

$$\boxed{\kappa(p) = \frac{\left| f''(x)\, |f'(x)|^2 - (f''(x) \,|\, f'(x))\, f'(x) \right|}{|f'(x)|^4}}$$

$p = f(x)$

this means moving perspective of traveler

$\boxed{\text{The moving frame}}$ (in $\mathbb{R}^3$)



$$T(p) = \frac{f'(x)}{|f'(x)|} = F'(s) \qquad N(p) = \frac{F''(s)}{|F''(s)|} = \frac{f''(x)\, |f'(x)|^2 - (f''(x)|f'(x))\, f'(x)}{|f'(x)|^4}$$

$B(p) = T(p) \times N(p)$ ← just cross product.

Review: Chainrule

Descartes / Newton

$$\frac{dx}{ds} \qquad \frac{ds}{dx}$$

Chainrule

$$\frac{dy}{ds} = \frac{dy}{dx}\,\frac{dx}{ds}$$

$$x =$$

$$1 = \frac{d}{dx}\,x = \frac{dx}{ds}\,\frac{ds}{dx}$$

$$0 = \frac{d}{dx}\left(\frac{dx}{ds}\cdot\frac{ds}{dx}\right)$$

$$= \frac{d^2x}{ds^2}\,\frac{ds}{dx}\,\frac{ds}{dx} + \frac{dx}{ds}\,\frac{d^2s}{dx^2}$$

Lagrange $\quad f(g(x)) = F$

$$F' = f'(g(x))\,g'(x)$$

$$F'' = f''(g(x))\,g'(x)^2 + f'(g(x))\,g''(x)$$

Apply this to $g : [a\,b] \to [c\,d]$ and

$f = g^{-1}$ the inverse function

We get $F(x) = x$ $\quad F = g^{-1}(g(x)) = x$

$$0 = F'' = (g^{-1})''(g(x))\, g'(x)^2 + (g^{-1})'(g(x))\, g''(x)$$

"More complicated" but important when calculating $(g^{-1})''$ at the correct point $g(x)$

<u>Example</u> $\quad g(x) = x^{1/3} \quad g: [0,1] \to [0,1]$

$$g^{-1}(y) = y^3 \qquad (g^{-1})' = 3y^2$$

$$g'(x) = \frac{1}{3} x^{-2/3} \quad g''(x) = \frac{-2}{9} x^{-5/3} \qquad (g^{-1})'' = 6y$$

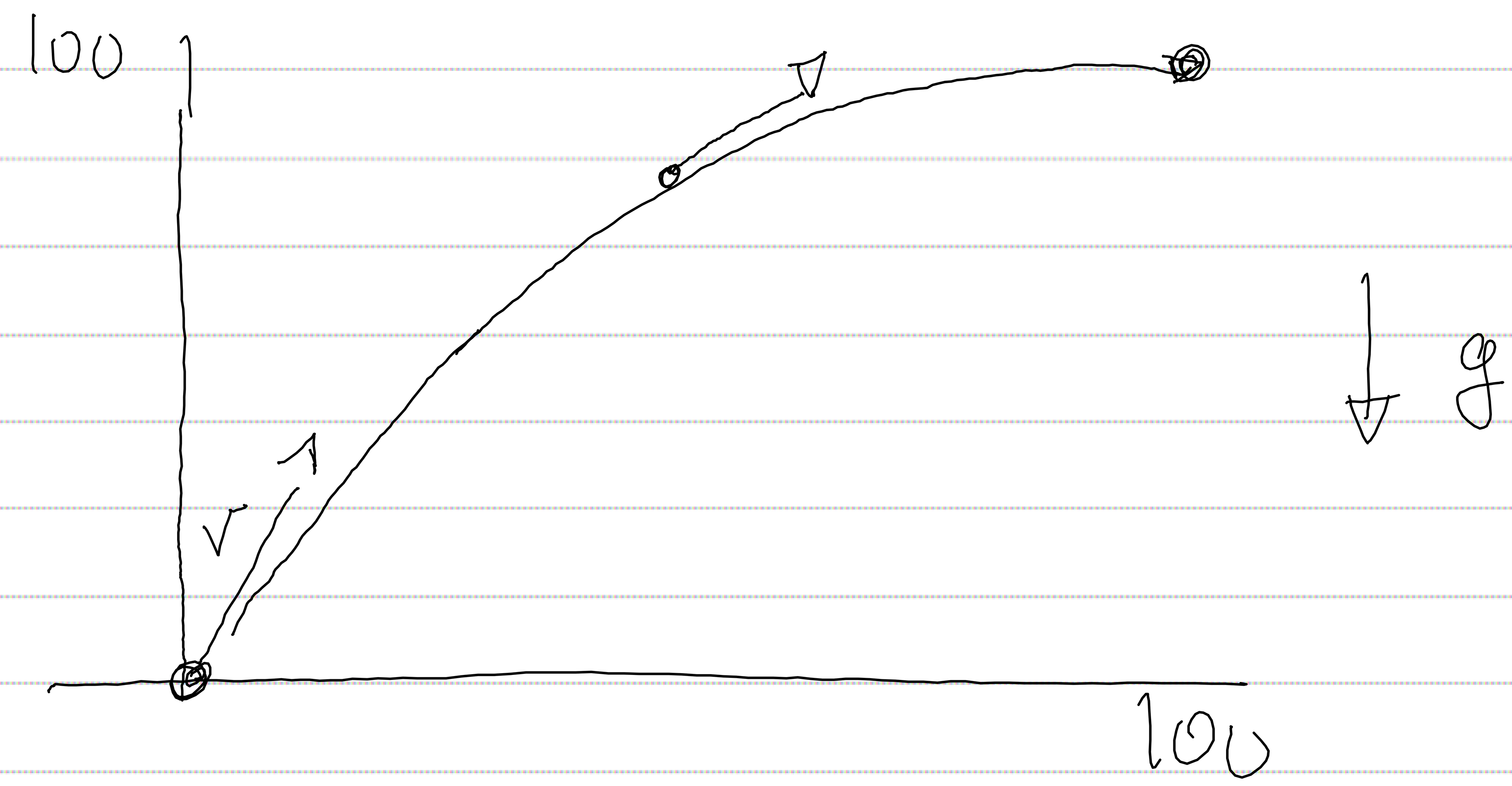
$$0 = 6y \left(\frac{1}{3} x^{-2/3}\right)^2 + 3y^2 \left(-\frac{2}{9}\right) x^{-5/3}$$

<u>where</u> $\boxed{y = g(x) = x^{1/3}}$

don't forget

Pb Starting point $(0,0)$ $m = \text{mass}$

Shoot a rocket to $(100, 100)$



Needed initial velocity

Position function

$r(t)$ $r(0) = (0,0)$

$r'(0) = v$

$r'(t)$ velocity $F = ma$

$= mg$

$r''(t) = (0, g\cancel{m})$

$r'(t) = (0, g\cancel{m}t) + (v_0, v_1)$

$r(t) = (v_0 t, v_1 t + g\cancel{m}\frac{t^2}{2})$

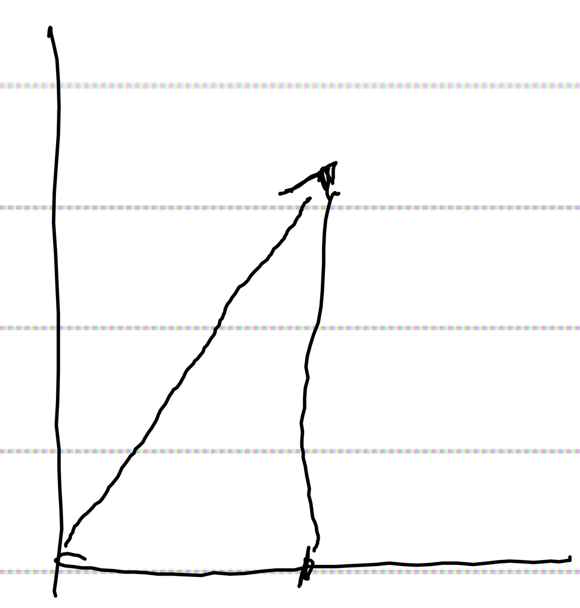
$$\cancel{(100, 100) = \left(v_0 t, v_1 t + \frac{gm}{2} t^2\right)}$$

$$\cancel{t = \frac{100}{v_0}}$$

$$\cancel{100 = v_1 t + \frac{gm}{2}\left(\frac{100}{v_0}\right)^2}$$

$$\cancel{= \frac{v_1}{v_0} 100 + \frac{gm}{2} \frac{100^2}{v_0^2}}$$

$$\cancel{100 v_0^2 = 100 v_1 v_0 + \frac{gm}{2} 100^2}$$



$v_0 = 1$

$$\boxed{v_0 = 1}$$

$$x = v_0 t \qquad \frac{x}{v_0} = t$$

$$y = v_1 t - \frac{gt^2}{2} = \frac{v_1}{v_0} x - \frac{g}{2} \frac{x^2}{v_0^2}$$

$$100 = v_1 100 - \frac{g}{2} 100^2$$

The curve is given by

$$v_1 - 1 = \left(\frac{g}{2}\right) 100$$

$$C = \left\{ \left(x, \frac{v_1}{v_0} x - \frac{g}{2} \frac{x^2}{v_0^2}\right) : x \in \mathbb{R} \right.$$

$$\boxed{v_1 = 1 + \frac{g}{2} 100}$$

$$\boxed{v_0 = 1}$$

$$(100, 100) = \left(x, \frac{v_1}{v_0} x - \frac{g}{2} \frac{x^2}{v_0^2}\right) \text{ means } x = 100$$

$$y = \frac{v_1}{v_0} 100 - \frac{g}{2} \frac{100^2}{v_0^2} = 100$$

$$100 v_0^2 = v_0 v_1 100 - \frac{g}{2} 100^2$$

$$v_0^2 - v_0 v_1 + \left(\frac{v_1}{2}\right)^2 = \frac{g}{2} 100 + \left(\frac{v_1}{2}\right)^2$$

$$\left(v_0 - \frac{v_1}{2}\right)^2 = \frac{g}{2} 100 + \left(\frac{v_1}{2}\right)^2$$

$$\Downarrow$$

determines angle