Calculus for functions on $\mathbb{R}^n$

Recall $\quad f : [a,b] \longrightarrow \mathbb{R}^d$

"parametisation for curve"

Now "parametisation for surface".

Ex $\quad F(r,s) = \begin{pmatrix} 1 + r + s \\ 2 + 3r + 4s \\ 3 + r \end{pmatrix}$ $\qquad \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$

$$F(\mathbb{R}^2) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \underset{-4x + y + z = 2}{\overset{-4 \quad +2 \quad +3}{}} \right\}$$

Definition $\;$ Let $\Omega \subseteq \mathbb{R}^d$ be a set. A function

$F : \Omega \longrightarrow \mathbb{R}^m$ is called continuous at $p \in \Omega$

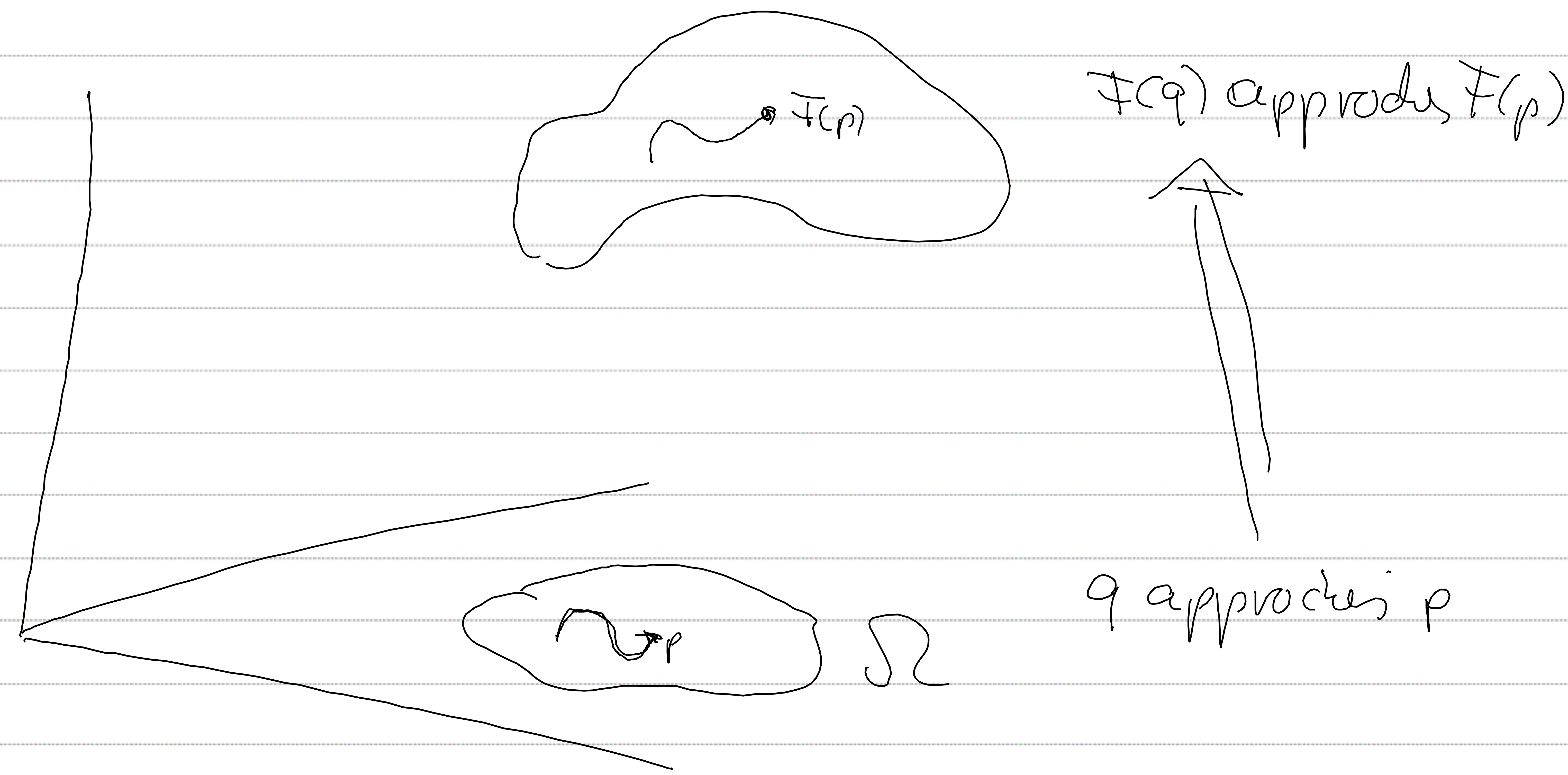
if

$$\boxed{\lim_{\substack{q \to p \\ q \in \Omega}} F(q) = F(p)}$$

or $\quad \lim_{q} |q_n - p| = 0 \implies \lim_{n} |F(q_n) - F(p)| = 0$

or $\quad$ For every $\varepsilon > 0$ there ex $\delta > 0$ such that

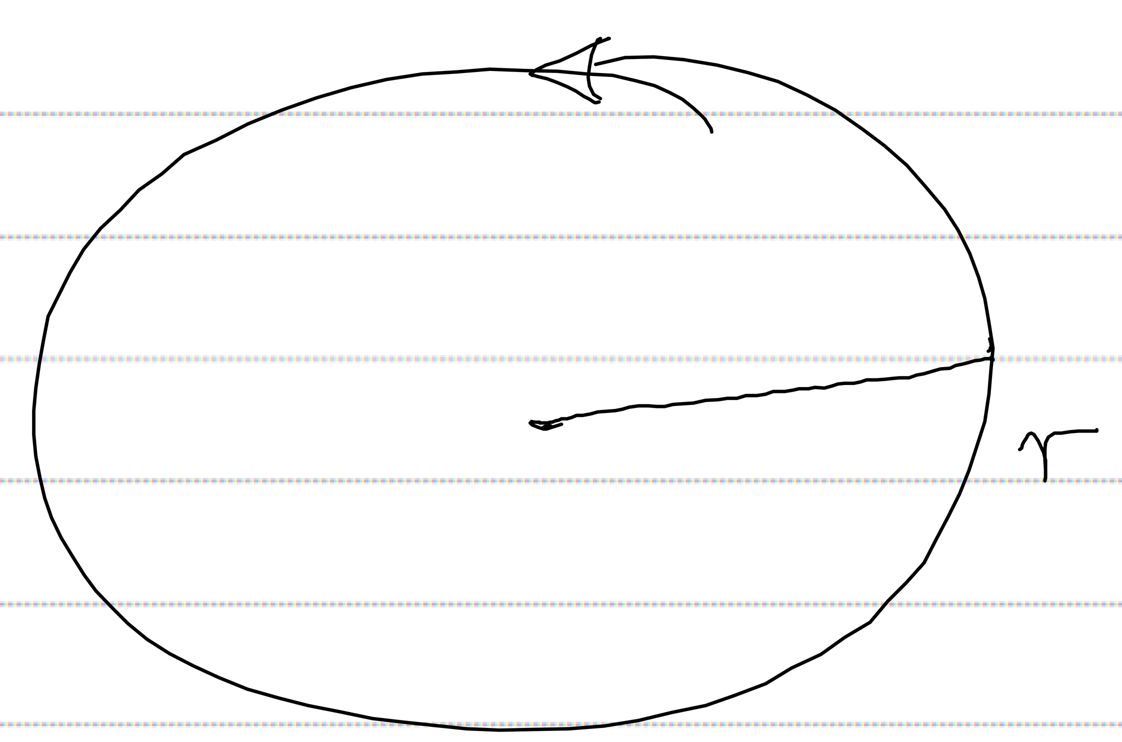
$|p-q|<\delta$ implies $|F(p)-F(q)|<\varepsilon$



Remark For $\Omega \subset \mathbb{R}$ this is continuity of vector valued functions

Example ~~$F(r,\theta)=$~~

$$F\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$



Claim F is continuous

Fix r_0, θ_0

$$|F(r,\theta) - F(r_0,\theta_0)| = \left\| \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} - \begin{pmatrix} r_0\cos\theta_0 \\ r_0\sin\theta_0 \end{pmatrix} \right\|$$

$$\leq \left\| \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} - \begin{pmatrix} r_0\cos\theta \\ r_0\sin\theta \end{pmatrix} \right\| + \left\| \begin{pmatrix} r_0(\cos\theta - \sin\theta_0 \\ r_0\sin\theta - \sin\theta_0 \end{pmatrix} \right\|$$

$$\leq |r-r_0| + |r_0| \max_{\theta \leq \tilde{\theta} \leq \theta_0} \left\| \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\| |\theta - \theta_0|$$

$$\leq |r-r_0| + |r_0|\,|\theta-\theta_0|$$

Given $\varepsilon > 0$ we choose

$$\delta = \min\left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2|r_0|}\right)$$

Then $|(r,\theta) - (r_0,\theta_0)| < \delta$ implies $|r-r_0| < \varepsilon/2$

and $|\theta-\theta_0| < \frac{\varepsilon}{2|r_0|}$ and hence

$$|F(r,\theta) - F(r_0,\theta_0)| < \varepsilon.$$

Practical criterion

$$F\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix} f_1(x_1 \dots x_n) \\ \vdots \\ f_m(x_1, \dots x_n)\end{pmatrix}$$

Above

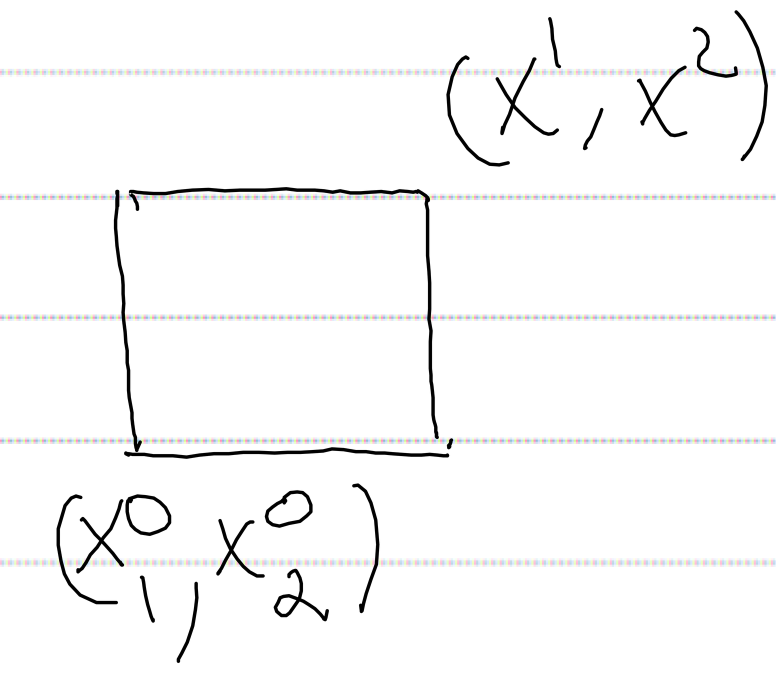
$$F\binom{r}{\theta} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} f_1(r,\theta) \\ f_2(r,\theta)\end{pmatrix}$$

Def $\frac{\partial f_j}{\partial x_i}(x) \cancel{\text{is}} = \lim_{h\to 0} \frac{f_j(x_1,\dots,x_i+h,\dots x_n) - f_j(x_1\dots x_n)}{h}$ is called partial derivative

Theorem $\max\limits_{|x-x_0|} \left|\frac{\partial f_j}{\partial x_i}(x)\right| \le K$

Then $|F(x)-F(x_0)| \le nmK\,|x-x_0|$

Proof



$$|F(x_1 \dots x_n) - F(x_1^0 \dots x_n^0)|$$

$$\le \sum_{j=1}^{m} |f_j(x_1 \dots x_n) - f_j(x_1^0 \dots x_n^0)|$$

$$\le \sum_{j=1}^{m}\sum_{i=1}^{n} |f_j(x_1 \dots x_i, x_{i+1}^0 \dots x_n^0) - f_j(x_1 \dots x_i^0, x_{i+1}^0 \dots x_n^0)|$$

$$\le \sum_{j=1}^{m}\sum_{i=1}^{n} K|x_i - x_i^0| \le Kmn\left(\sum_{i=1}^{n}|x_i-x_i^0|^2\right)^{1/2} \quad \square$$

Recall

$f:(a,b) \to \mathbb{R}^d$ differentiable if

$$\frac{|f(t) - \overbrace{[f(t_0) + f'(t_0)(t-t_0)]}^{\text{"Line equation"}}|}{|t-t_0|} \longrightarrow 0$$

$$\mathbb{R}^k$$
$$\cup$$

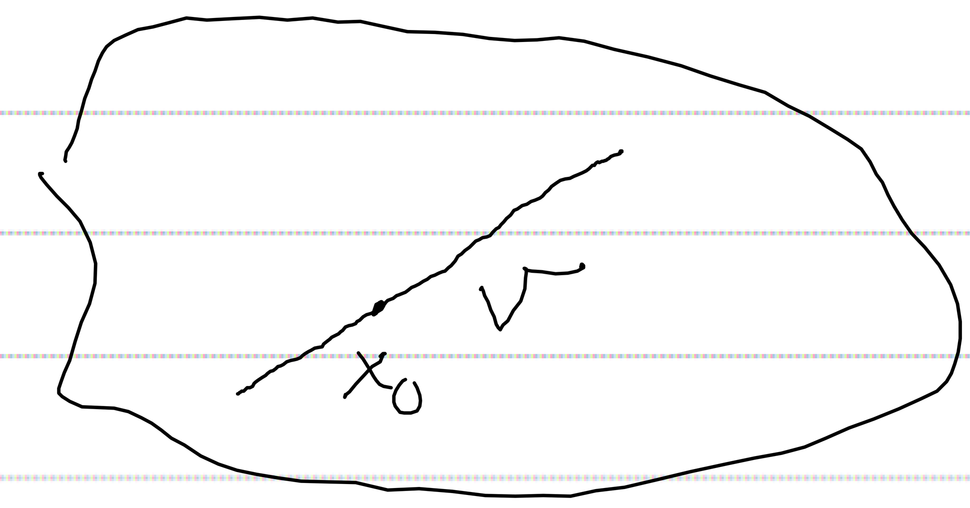
Definition $f: \mathrm{dom}(f) \longrightarrow \mathbb{R}^d$ is differentiable at x_0 if there exists a linear map $a: \mathbb{R}^k \longrightarrow \mathbb{R}^d$ such that

$$\boxed{\frac{\|f(x) - [f(x_0) + a(x-x_0)]\|}{\|x-x_0\|} \xrightarrow[x \to x_0]{} 0}$$

Then $f'(x_0) = a$ in the space of linear maps

Note $\{a: \mathbb{R}^k \to \mathbb{R}^d$: a linear $\}$ is also a vector space. And we may put an inner product

$$(a|b) = \sum_{ij} a_{ij} b_{ij}$$

Observation $dom(f) =$ 

Let $x_0 \in dom(f)$ such that f is differentiable at x_0
$v \in \mathbb{R}^k$ a vector

$\{x_0 + tv : -\varepsilon_0 \leq t \leq \varepsilon_0\} \subset dom(f)$.

The $\mathbb{R}^d$ valued function

$$g(t) = f(x_0 + tv)$$

is also differentiable at 0 (Mini chain rule)

$$\boxed{g'(0) = f'(x_0)(v)}$$

Proof:

$$\frac{|g(t) - g(0) - g'(0)t|}{|t|}$$

$$= \frac{|f(\overbrace{x_0 + tv}^{x}) - f(x_0) - \cancel{t}a(\cancel{t}v)|}{|tv|}|v|$$

$$= \frac{|f(x_0 + t) - [f(x_0) + a(x_0 + tv - x_0)]|}{|tv|}|v|$$

$$\xrightarrow[t \to 0]{} 0$$

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