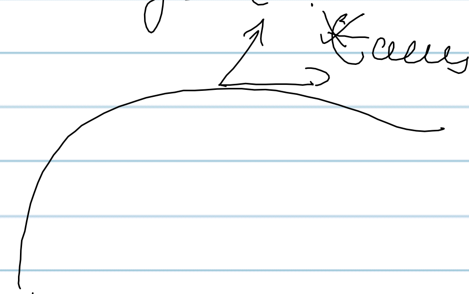
Differential forms

(1) We have seen the line integral



$$\int_C X \cdot ds$$

length measure

$$= \int_C X \cdot t(x)\, d\sigma(x)$$

tangent vector.

as calculating the "average interaction of a vector field with the tangent vector".

Similar notions exist for higher dimensional objects.

(2) The motivation for the general form of Stokes theorem

$$\int_M d\omega = \int_{\partial M} \omega$$

M oriented manifold

is the Fundamental theorem of calculus

$$\int_a^b f'(x) = f(b) - f(a)$$

③ What are differential forms?

<u>First answer</u>

Formal linear combinations of the form

$f\, d\omega_1 + g\omega_2 + h\omega_3 \dots$

where f, g, h are C^1 functions.

and

$\omega_1 \in \{dx, dx \wedge dy, dx \wedge dy \wedge dz\}$

are symbols

<u>Rules</u> 1) $(f dx) \wedge dy = f(dx \wedge dy) = dx \wedge (f dy)$

2) $dx \wedge dy = -dy \wedge dx$

3) $dx \wedge (dy \wedge dz \dots) = dx \wedge dy \wedge dz \dots$

These rules are made to make a change of variable easier

$$\int_{\Phi(D)} F \, dx \wedge dy = \int_D F \circ \Phi \, \det \Phi'(\alpha, \beta) \, d\alpha \wedge d\beta$$

Here is how this works

$$x = x(\alpha, \beta)$$
$$y = y(\alpha, \beta)$$

$$dx \wedge dy = dx(\alpha, \beta) \wedge dy(\alpha, \beta)$$

$$= \left(\frac{\partial x}{\partial \alpha} d\alpha + \frac{\partial x}{\partial \beta} d\beta\right) \wedge \left(\frac{\partial y}{\partial \alpha} d\alpha + \frac{\partial y}{\partial \beta} d\beta\right)$$

$$= \left(\frac{\partial x}{\partial \alpha} \frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \beta} \frac{\partial y}{\partial \alpha}\right) d\alpha \wedge d\beta$$

(used $d\alpha \wedge d\alpha = - d\alpha \wedge d\alpha \Rightarrow d\alpha \wedge d\alpha = 0$
$d\beta \wedge d\beta = 0$

$d\alpha \wedge d\beta = - d\beta \wedge d\alpha$)

In this calculation we see that

$$d(fw) = \frac{\partial f}{\partial x} dx \wedge w + \frac{\partial f}{\partial y} dy \wedge dw + \cdots$$

is given by the so called total differential.

(meaning we analyze f and w according to all hidden variables (given by change of variable)).

Second answer

For a fixed dimension d the differential forms of order k form a vector space of C^∞ functions

$d=1 \quad k=0 \quad Df^0(\mathbb{R}) = C_\infty(\mathbb{R})$

$d=1 \quad k=1 \quad Df^1(\mathbb{R}) = C_\infty(\mathbb{R})$

It seems this are the same spaces, but they serve a different purpose !

A differential form of order k
combined with a manifold of dimension k
can be combined with the integral
and gives a number

$$\int_M \omega \in \mathbb{R}$$

Example

$$\omega = f\,dx \qquad \text{in } \mathbb{R}$$

$$\int_{[a,b]} \omega = \int_a^b f(x)\,dx \qquad \text{is the integral}$$

$$\int_{\{a\}} f = f(a) \qquad \text{strange}$$

but now

$$\int_{[a,b]} df = \int_{[a,b]} f'(x)\,dx = f(b) - f(a)$$

-1 a $+1$ b

$$= \oint_{\{a\} \cup \{b\}} f \qquad \text{Here } \oint \text{ means oriented}$$

For $d=2$ we have

$k=0$ $\quad \mathcal{D}f^0(\mathbb{R}^2) = C_\infty(\mathbb{R}^2)$

$k=1$ $\quad \mathcal{D}f^1(\mathbb{R}^2) = C_\infty(\mathbb{R}^2) \times C_\infty(\mathbb{R}^2) = C_\infty(\mathbb{R}^2)^2$

$k=2$ $\quad \mathcal{D}f^2(\mathbb{R}^2) = C_\infty(\mathbb{R}^2)$

In order to distinguish ~~k~~ $k=0$ and $k=2$ we write

$$f \in \mathcal{D}f^0(\mathbb{R}^2)$$

$$f\,\underbrace{(dx\wedge dy)}_{\text{two }d = \text{order } 2} \in \mathcal{D}f^2(\mathbb{R}^2)$$

Finally $\quad f\,dx + g\,dy \in \mathcal{D}f^1(\mathbb{R}^2)$

arbitrary f $\quad$ arbitrary g $\longrightarrow$ two copies of $C_\infty(\mathbb{R}^2)$

Integrals

$k=0$ $\quad \int_a f = f(a)$

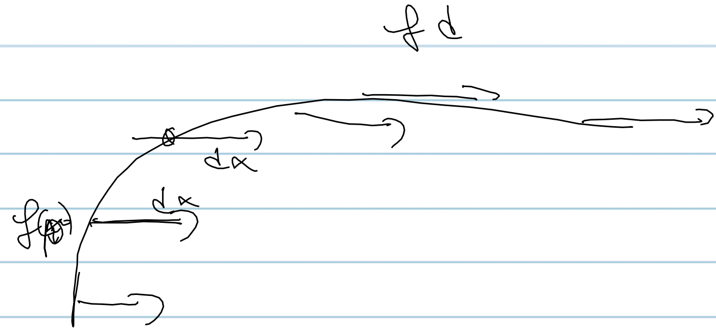
$k=2$ $\quad \int_D f\,dx\wedge dy = \int_D f\,d(x,y) \Leftarrow$ area integral with cont function f

usually

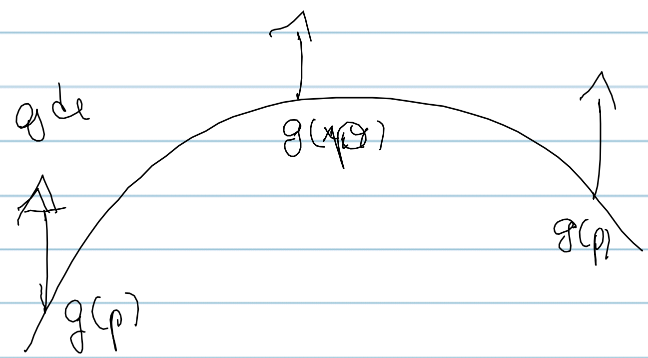
$k=1 \qquad \int_C f\,dx + g\,dy = \int_C \begin{pmatrix} f \\ g \end{pmatrix} \cdot ds$

View f, g as <u>coefficients of a vector field</u>!

geometric calc



C curve



<u>Green's theorem</u>: let D be an oriented domain with closed curve ∂D as boundary

Then

$$\boxed{\int_D d\omega = \int_{\partial D} \omega}$$

This is a good memo technical trick

$$\int_{\partial D} \underline{X} \cdot t \, d\sigma = \int_{\partial D} f\,dx + g\,dy$$

$$X = \begin{pmatrix} f \\ g \end{pmatrix} \underset{\text{Green}}{=} \int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$$

$$= \int_D \left(\frac{\partial X_2}{\partial x} - \frac{\partial X_1}{\partial y} \right) dx \wedge dy$$

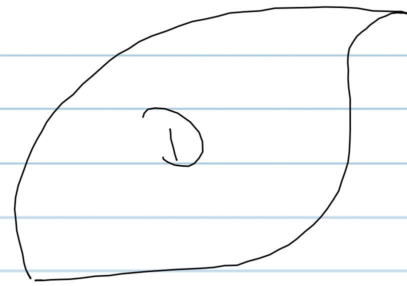
Application 1) Let $\underline{X}$ be a vector field satisfying the Cauchy-Riemann equations

$$\frac{\partial X_2}{\partial x} = \frac{\partial X_1}{\partial y}$$

Let Ω be a simply connected domain and $C \subset \Omega$ a closed path.

Then $\oint_C X \cdot ds = 0$

Indeed



closed path

includes a reasonable Domain D

oriented and closed

$$\oint_C X\, ds = \int_D \Big(\overbrace{\frac{\partial X_2}{\partial x} - \frac{\partial X_1}{\partial y}}^{=0}\Big)\, d(x,y) = 0$$

Hence $\oint_C X\cdot ds \cancel{\text{is}} = 0$

This allows us to define

$$F(\vec{x}) = F(\vec{x_0}) + \int_{\vec{x_0}}^{\vec{x}} X\cdot ds$$

for any given point and this definition does not depend on the choice of the smooth path from $\underset{\text{vectors}}{\vec{x_0}} \longrightarrow \underset{\text{vector.}}{\vec{x}}$

Then it is easy to show that

$$\boxed{\nabla F = X}$$

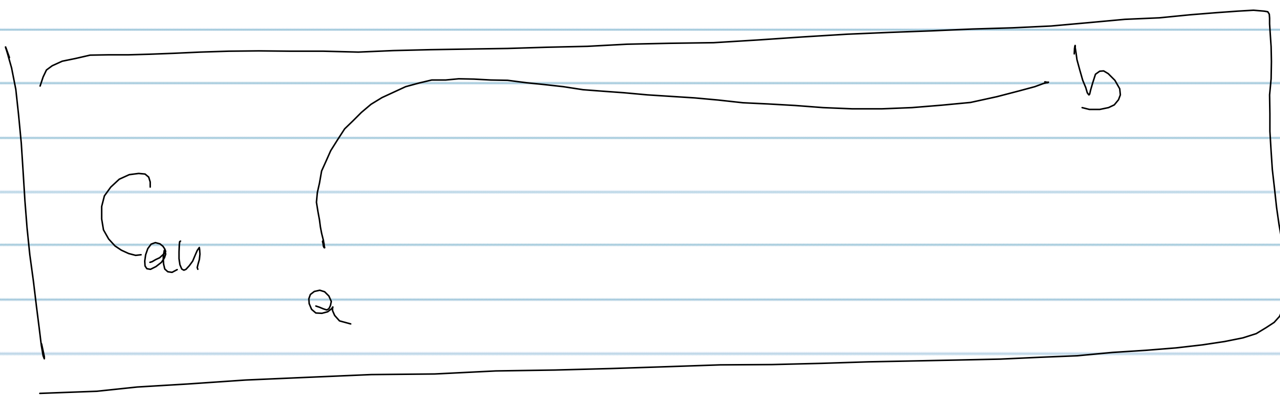
Hence X has a potential

Application 2 If X satisfies $\frac{\partial X_2}{\partial x} = \frac{\partial X_1}{\partial y}$)

then The potential F corresponds to the work need to move a particle from a point $a \in \mathbb{R}^2$ to a point $b \in \mathbb{R}^2$

$$\boxed{F(b) - F(a) = \int_a^b \overline{X} \cdot ds}$$

$$= \int_{C_{ab}} \overline{X} \cdot ds$$



any curve (again some kind of fundamental theorem)

Dimension $d = 3$

Here we have

$$\mathcal{D}f^0(\mathbb{R}^3) = C_0(\mathbb{R}^3)$$
$$\mathcal{D}f^1(\mathbb{R}^3) = C_0(\mathbb{R}^3)^3$$
$$\mathcal{D}f^2(\mathbb{R}^3) = C_0(\mathbb{R}^3)^3$$
$$\mathcal{D}f^3(\mathbb{R}^3) = C_0(\mathbb{R}^3)$$

By the way in general

$$\mathcal{D}f^k(\mathbb{R}^d) = C_0(\mathbb{R}^d)^{\binom{d}{k}}$$ binomial coefficient

$$\begin{array}{c} 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \end{array}$$

$\binom{d}{k}$ given by Pascal triangle

$k=0$ $\mathcal{D}f^0(\mathbb{R}^3)$ $\quad \int_{\{a\}} f = f(a)$ functions evaluate at a point

$k=1$ $\quad \Omega^1(\mathbb{R}^3) = \{ f\,dx + g\,dy + h\,dz \mid f, g, h \in C^\infty(\mathbb{R}^3) \}$

<u>Integration</u>

$$\int_C f\,dx + g\,dy + h\,dz = \int_C \begin{pmatrix} f \\ g \\ h \end{pmatrix} \cdot ds$$

again given by our line integral

<u>Remark</u> We may again talk about ~~for~~ work to transport a point in a vector field

$$F(b) - F(a) = \int_{C_{ab}} X \cdot ds$$

But that may depend on connecting path unless X is <u>conservative</u> $=$

$$X = \nabla F \text{ for some } F$$

$\underline{k=2}$ $\quad \mathrm{Diff}^2(\mathbb{R}^3) = C_0(\mathbb{R}^3)^3$

$= \{ f\, dx \wedge dy + g\, dy \wedge dz + h\, dz \wedge dx \,;\, f, g, h \in C^\infty(\mathbb{R}^3) \}$

We need to two d's to integrate a surface!

How do we do that?

$\underline{M} = \underline{\Phi}(D) \qquad D \subset \mathbb{R}^2$ is

a parametrisation.

We follow the total differential approach

$x = x(\alpha, \beta) \qquad y = y(\alpha, \beta) \qquad z = z(\alpha, \beta)$

$$\underline{\Phi}(\alpha, \beta) = \begin{pmatrix} x(\alpha, \beta) \\ y(\alpha, \beta) \\ z(\alpha, \beta) \end{pmatrix}$$

$$dx \wedge dy = \left(\frac{\partial x}{\partial \alpha} d\alpha + \frac{\partial x}{\partial \beta} d\beta \right) \wedge \left(\frac{\partial y}{\partial \alpha} d\alpha + \frac{\partial y}{\partial \beta} d\beta \right)$$

$$= \left(\frac{\partial x}{\partial \alpha} \frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \beta} \frac{\partial y}{\partial \alpha} \right) d\alpha \wedge d\beta$$

In ~~general~~ general

$\omega \circ \Phi$

$= f\, dx \wedge dy + g\, dy \wedge dz + h\, dz \wedge dx \circ \Phi$

$$= f \circ \Phi \left(\frac{\partial x}{\partial \alpha}\frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \beta}\frac{\partial y}{\partial \alpha} \right) d\alpha \wedge d\beta$$

$$+ g \circ \Phi \left(\frac{\partial y}{\partial \alpha}\frac{\partial z}{\partial \beta} - \frac{\partial y}{\partial \beta}\frac{\partial z}{\partial \alpha} \right) d\alpha \wedge d\beta$$

$$+ h \circ \Phi \left(\frac{\partial z}{\partial \alpha}\frac{\partial x}{\partial \beta} - \frac{\partial z}{\partial \beta}\frac{\partial x}{\partial \alpha} \right) d\alpha \wedge d\beta$$

Therefore:

$$\int_M \omega = \int_D \left[(f \circ \Phi)\left(\frac{\partial x}{\partial \alpha}\frac{\partial y}{\partial \beta} - \right) \cdots \right] d\alpha \wedge d\beta$$

Is given by an area integral !

Again is good to have a memo technical trick to remember that !

Recall the cross product

$$\frac{\partial \Phi}{\partial \alpha} \times \frac{\partial \Phi}{\partial \beta} = \begin{pmatrix} \frac{\partial x}{\partial \alpha} \\ \frac{\partial y}{\partial \alpha} \\ \frac{\partial z}{\partial \alpha} \end{pmatrix} \times \begin{pmatrix} \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial \beta} \\ \frac{\partial z}{\partial \beta} \end{pmatrix}$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\ \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \end{vmatrix} = \begin{pmatrix} \frac{\partial y}{\partial \alpha}\frac{\partial z}{\partial \beta} - \frac{\partial y}{\partial \beta}\frac{\partial z}{\partial \alpha} \\ \frac{\partial z}{\partial \alpha}\frac{\partial x}{\partial \beta} - \frac{\partial x}{\partial \alpha}\frac{\partial z}{\partial \beta} \\ \frac{\partial x}{\partial \alpha}\frac{\partial y}{\partial \beta} - \frac{\partial y}{\partial \alpha}\frac{\partial x}{\partial \beta} \end{pmatrix}$$

$$\boxed{\text{We get } \omega \circ \Phi = \begin{pmatrix} h \circ \Phi \\ g \circ \Phi \\ f \circ \Phi \end{pmatrix} \cdot \frac{\partial \Phi}{\partial \alpha} \times \frac{\partial \Phi}{\partial \beta}}$$

Since $Df^1(\mathbb{R}^3) = C_\infty(\mathbb{R}^3)^3$ and $Df^2(\mathbb{R}^3) = C_\infty(\mathbb{R}^3)^3$ have the same dimension 3 over $C_\infty(\mathbb{R}^3)$ we can turn $\omega \in Df^2(\mathbb{R}^3)$ to a form

$$\tilde{\omega} = f\,dz + g\,dx + h\,dz \mathrel{\hat{=}} \begin{pmatrix} g \\ h \\ f \end{pmatrix}$$

and read it as a vector field

$$X_\omega = \begin{pmatrix} h \\ g \\ f \end{pmatrix}$$

comment: It is nice to have a vector field making this computation easier: Reason (3)

Then we get

normal vector

area measure

$$\boxed{\int_{\Pi} \omega = \int_{\Pi} X_\omega \cdot n \; d\sigma}$$

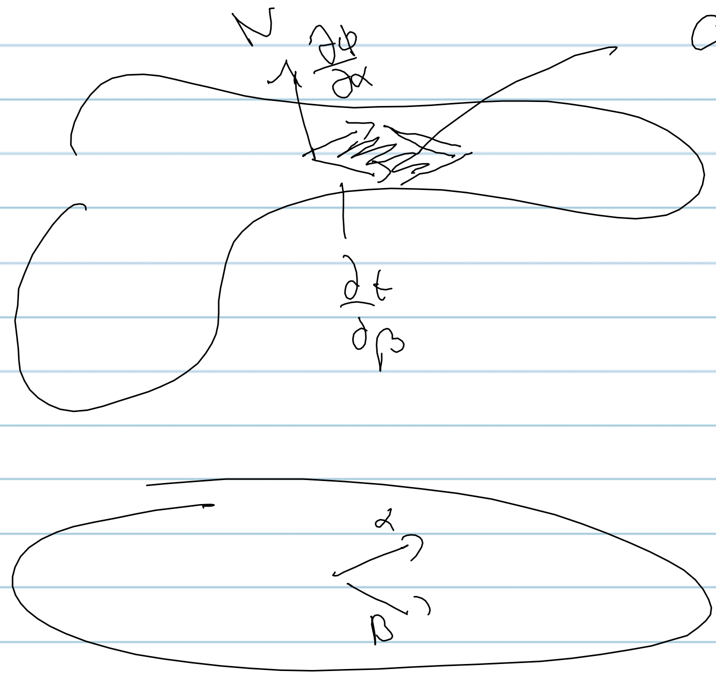
Note that

$$n = \frac{\frac{\partial \phi}{\partial \alpha} \times \frac{\partial \phi}{\partial \beta}}{\left| \frac{\partial \phi}{\partial \alpha} \times \frac{\partial \phi}{\partial \beta} \right|}$$

and $$\int_{\Pi} F(x)\, d\sigma(x) = \int_D F \circ \Phi \, \left|\det \Phi'(\alpha,\beta)^t \, \Phi'(\alpha,\beta)\right|^{1/2} d(\alpha,\beta)$$

and $$\left|\det \Phi'(\alpha,\beta)^t \, \Phi'(\alpha,\beta)\right|^{1/2} = \left| \frac{\partial \phi}{\partial \alpha} \times \frac{\partial \phi}{\partial \beta} \right| \quad (3)$$

Equality (3) can be explained geometrically



area des tenten

area $\langle \text{du} \rangle$

$= |N|$

$= \boxed{\text{length of normal vector}}$

$= \left| \frac{\partial \phi}{\partial \alpha} \times \frac{\partial \phi}{\partial \beta} \right|$

Geometric interpretation

(1) A differential form of order 2 on $\mathbb{R}^3$ has general form

$$\omega = f\, dx \wedge dy + h\, dy \wedge dz + g\, dz \wedge dx$$

but can be expressed with respect to any orthonormal basis of $\mathbb{R}^3$

$$= \hat{f}\, d\xi_1 \wedge d\xi_2 + \hat{h}\, d\xi_2 \wedge d\xi_3 + \hat{g}\, d\xi_3 \wedge d\xi_1$$

We may choose ξ_1, ξ_2 to be the basis of the tangent space at a certain ~~fem~~ point and $\xi_3 = n$

Then $\displaystyle\int_M \omega = \int_\Phi (\omega \circ \Phi)\, d\alpha \wedge d\beta$

only requires knowledge of the component

$\boxed{f\, d\xi_1 \wedge d\xi_2}$ at point $p = \Phi(\alpha, \beta)$

Indeed

$$\omega \circ \Phi = X_\omega \cdot \left(\frac{\partial \phi}{\partial \alpha} \times \frac{\partial \phi}{\partial \beta} \right)$$

$$= X_\omega \cdot \xi_1 \times \xi_2 \quad |N|$$

$$= X_\omega \cdot \xi_3 \quad |N|$$

$$= X_\omega \cdot N$$

but

$$\omega = \hat{f}\, d\xi_1 \wedge d\xi_2 + \hat{h}\, d\xi_2 \wedge d\xi_3 + \hat{g}\, d\xi_3 \wedge d\xi_1$$

gives

$$\underline{X}_{\hat{\omega}} = \begin{pmatrix} \hat{g} \\ \hat{h} \\ \hat{f} \end{pmatrix}$$

$$\underline{X}_{\hat{\omega}} \cdot N = \begin{pmatrix} \hat{g} \\ \hat{h} \\ \hat{f} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot |N|$$

$$\boxed{= \hat{f}\, |N|}$$

This means we "ignore" the $d\xi_2 \wedge d\xi_3$ and $d\xi_3 \wedge d\xi_1$ components.

In vector space terminology this corresponds to projecting ω onto the $d\xi_1 \wedge d\xi_2$ component.

$$\boxed{\langle \omega\ d\xi_1 \wedge d\xi_2 \rangle = X_\omega \cdot n}$$

Example:

$$\Phi(\alpha,\beta) = \begin{array}{cc} \begin{pmatrix} \alpha^2\beta \\ \beta^2 \\ \sqrt{\alpha+\beta^2} \end{pmatrix} & \begin{array}{c} = x \\ = y \\ = z \end{array} \end{array} \qquad M = \Phi([0,1]^2)$$

$$\omega = x^2 z \, dz$$

$$d\omega = 2xz \, dx \wedge dz$$

$$d\omega \circ \Phi = 2\alpha^2\beta \, \sqrt{\alpha+\beta^2} \; dx \wedge dz \circ \Phi$$

$$= 2\alpha^2\beta\sqrt{\alpha+\beta^2} \left(\frac{2\alpha\beta\beta}{\sqrt{\alpha+\beta^2}} - \frac{\alpha^2}{2\sqrt{\alpha+\beta^2}} \right) d\alpha \wedge d\beta$$

$$= (4\alpha^3\beta^3 - \alpha^4\beta) \, d\alpha \wedge d\beta$$

$$\int_M d\omega = \int_{[0,1]^2} (4\alpha^3\beta^3 - \alpha^4\beta) \, d(\alpha,\beta)$$

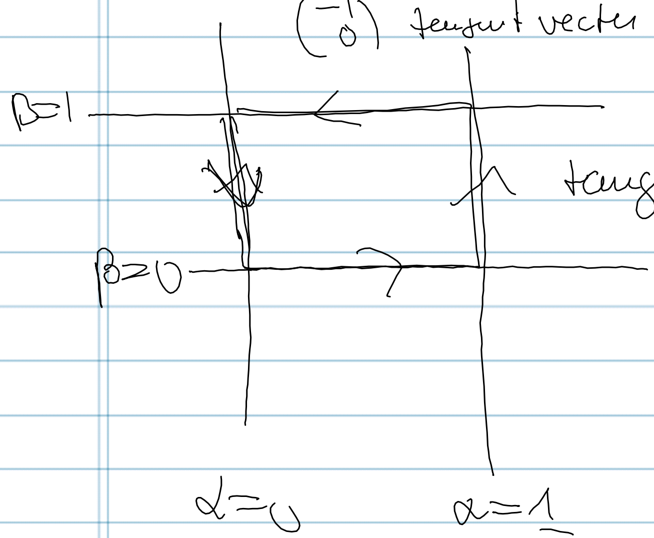
$$= 4 \, \frac{1}{4} \cdot \frac{1}{4} - \frac{1}{5}\frac{1}{2} = \frac{1}{4} - \frac{1}{10} = \frac{1}{20}$$

$$w = x^2 z\, dz$$

$$= \alpha^4 \beta^2 \sqrt{\alpha+\beta^2}\left(\frac{\partial z}{\partial \alpha} d\alpha + \frac{\partial z}{\partial \beta} d\beta\right)$$

$$= \alpha^4 \beta^2 \sqrt{\alpha+\beta^2}\left(\frac{d\alpha}{2\sqrt{\alpha+\beta^2}} + \frac{\beta}{\sqrt{\alpha+\beta^2}} d\beta\right)$$

$$= \frac{\alpha^4 \beta^2}{2} d\alpha + \alpha^4 \beta^3\, d\beta$$



$\alpha=0$ or $\beta=0$

gives no contribution

$$\int_{\partial \square} w \circ \phi = \int_0^1 \beta^3\, d\beta - \int_0^1 \frac{\alpha^4}{2} d\alpha$$

$$= \frac{1}{4} - \frac{1}{10} = \frac{1}{20}$$

works!

Final ~~ly~~ Remark

The map d = total differential maps

$$d : Df^k(\mathbb{R}^n) \longrightarrow Df^{(k+1)}(\mathbb{R}^n)$$

d makes degree of differential form larger by one

The operat $\partial M \longrightarrow \partial M$

reduces the dimension by one

Therefore Stokes theorem

$$\int_M d\omega = \int_{\partial M} \omega$$ Mounted manifold

makes sense