Gauss or Divergence theorem.

Thm Let $V \subset \mathbb{R}^3$ be a solid with positive orientation of ∂V = the boundary of V. Then

$$\int_{\partial V} \omega = \int_V d\omega$$

holds for every 2 form ω.

Divergence? Let ω be a two form.

$$\omega = f\, dx\wedge dy + g\, dy\wedge dz + h\, dz\wedge dx$$

Then

$(-1)(-1)=1$ $\qquad\qquad$ $(-1)(-1)=1$

$$d\omega = \frac{\partial f}{\partial z}\, dz\wedge dx\wedge dy + \frac{\partial g}{\partial x}\, dx\wedge dy\wedge dz + \frac{\partial h}{\partial y}\, dy\wedge dz\wedge dx$$

$$= \frac{\partial f}{\partial z}\, dx\wedge dy\wedge dz + \frac{\partial g}{\partial x}\, dx\wedge dy\wedge dz + \frac{\partial h}{\partial y}\, dx\wedge dy\wedge dz$$

Hence for $X = \begin{pmatrix} g \\ h \\ f \end{pmatrix}$ we find

$$d\omega_X = \operatorname{div}(X) \cdot dx\wedge dy\wedge dz$$

And the divergence of

$$\underline{X} = \begin{pmatrix} g \\ h \\ f \end{pmatrix}$$

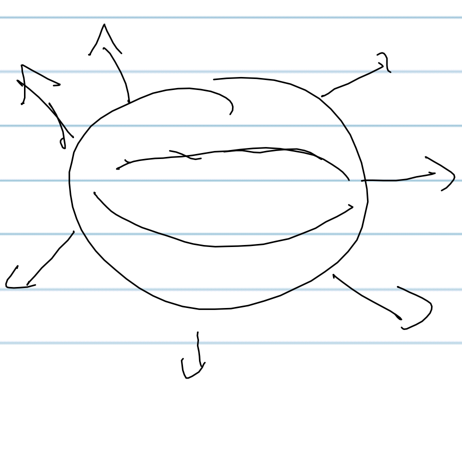
is given by

$$\operatorname{div}(\underline{X}) = \left(\frac{\partial g}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial f}{\partial z}\right)$$

everybody is differentiated after his <u>coordinate</u>

<u>Application 1</u>: Let $\underline{X}$ be a vector field with $\operatorname{div}(\underline{X}) = 0$

Then $$\boxed{\oint_{\partial V} (\underline{X} \cdot n)\, d\sigma = 0}$$



Divergence free means <u>no sources</u>

$\boxed{\text{Everything which comes in goes out}}$

Remark $\oint X\cdot n\, d\sigma$

$\int X\cdot t\, d\sigma$

was discovered through physics

Application 1b)

$$\operatorname{div} X(p) = \lim_{r\to 0} \frac{1}{\frac{4}{3}\pi r^3} \oint_{\partial B(p,r)} (\underline{X}\cdot n)\, d\sigma$$

Indeed

$$\lim_{r\to 0} \int_{B(p,r)} \operatorname{div}(\underline{X}) \frac{d\sigma}{\operatorname{vol}(B(p,r))} \Rightarrow \operatorname{div}(\underline{X})$$

provided $\operatorname{div}(\underline{X})$ is continuous

compare with $\boxed{n\cdot \operatorname{curl} X\, \underline{X}(p) = \lim_{r\to 0} \frac{1}{4\pi r^2} \int_{\partial C(0,r)} \underline{X}\cdot ds}$

Application 2 Gauss' formulae

$$(1) \quad \int_V f\Delta g + \int_V \nabla f\cdot \nabla g = \oint_{\partial V} f(\nabla g\cdot n)\, d\sigma$$

Proof $\underline{X} = f\cdot \nabla g$

$$\operatorname{div}(\underline{X}) = \left(\frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial z}\right)\right)$$

$$= \nabla f\cdot \nabla g + f\Delta(g)$$

Recall $\boxed{\Delta(g) = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}}$

(1)' $$\int_V f\Delta g - \Delta f\, g = \oint_{\partial V} (f\nabla g - g\nabla f)\cdot n$$

In particular let f, g be functions which vanish on the boundary of V. Then

$$\boxed{\int_V f\,\Delta g = \int_V \Delta f\, g}$$

This is the starting point of the following famous article with title

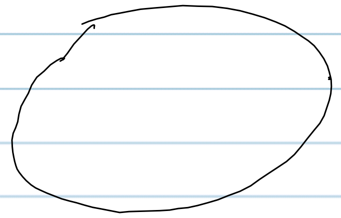
Can you here the shape of a drum.

Meaning for different domains V are the eigenvalues of Δ, i.e. those λ such that

$$\Delta(f) = \lambda f \qquad f \neq 0$$

<u>different?</u>

Idea.



$f|_{\partial D} = 0 \qquad \Delta(f) = 0$

drum

describe vibration of D

Lemma let $n \geq 3$ Then

$$\Delta\left(\frac{1}{|x|^{n-2}}\right) = 0 \qquad \text{for all } x \neq 0$$

Proof $f(x) = \left(\sum_{j=1}^{n} x_j^2\right)^{-\frac{n-2}{2}}$

$$\frac{\partial}{\partial x_j} f = -2x_j \frac{(n-2)}{2} \left(\sum_{j=1}^{n} x_j^2\right)^{-\frac{n-2}{2}-1}$$

$$= -(n-2) x_j \left(\sum_{j=1}^{n} x_j^2\right)^{-\frac{n}{2}}$$

$$\frac{\partial^2}{\partial x_j^2} f = -(n-2)\left(\sum_{j=1}^{n} x_j^2\right)^{-\frac{n}{2}} + (n-2)(n)\, x_j^2 \left(\sum_{j=1}^{n} x_j^2\right)^{-\frac{n+2}{2}}$$

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} f = -n(n-2)\left(\sum_{j=1}^{n} x_j^2\right)^{-\frac{n}{2}} + n(n-2)\left(\sum_{j=1}^{n} x_j^2\right)^{-\frac{n}{2}} = 0$$

<u>Lemma</u> let f be contuous

Then

$$\lim_{r\to 0} \oint_{\partial B(0,r)} f(x)\left(\nabla\left(\frac{1}{|x|}\right)\cdot n\right) d\sigma(x)$$

$$= -4\pi h(0)$$

<u>Proof</u> $\nabla\left(\frac{1}{|x|}\right) = -\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \frac{1}{|x|^3}$

$$\nabla\left(\frac{1}{|x|}\right)\cdot n = -\frac{x}{|x|^3}\cdot\frac{x}{|x|} = -\frac{|x|^2}{|x|^4} = \frac{1}{|x|^2}$$

$|x| = r$

$$\Rightarrow \nabla\left(\frac{1}{|x|}\right)\cdot n = \frac{1}{r^2}$$

$$|\partial B(0,r)| = 4\pi r^2$$

Thus $\int_{\partial B(0,r)} f(x) \frac{-1}{r^2} \triangleq f(0) \frac{-4\pi r^2}{r^2} = -f(0) 4\pi$

for f constant $\square$

Lemma $\lim_{r\to 0} \int_{\partial B(0,r)} \frac{\nabla f}{|x|} \cdot n \, d\sigma(x) = 0$

Proof We may assume $\nabla f = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ is constant

$$\int_{\partial B(0,r)} \frac{\langle \alpha, x \rangle}{|x|^2} \, d\sigma(x) \underset{\text{symmetry}}{=} \int \frac{\langle \alpha, -x \rangle}{|x|^2} \, d\sigma(x)$$

$$= -\int \frac{\langle \alpha, x \rangle}{|x|^2} \, d\sigma(x)$$

Thus for constant ∇f we get 0.

Actually it just remains to see that integral converges

$\alpha^2+\beta^2+z^2 = r^2$

$z = \sqrt{\ }$

$$\int_{\partial B(0,r)} \frac{\nabla f}{|x|} \cdot \frac{x}{|x|} \, d\sigma(x)$$

$$\phi\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \sqrt{r^2-(\alpha^2+\beta^2)} \end{pmatrix}$$

$$= \frac{1}{r^2} \int_{rD} (\nabla f \circ \phi) \cdot \begin{pmatrix} \alpha \\ \beta \\ \sqrt{r^2-\alpha^2-\beta^2} \end{pmatrix} \frac{r}{\sqrt{r^2-\alpha^2-\beta^2}} \, d(\alpha\beta)$$

$\int_0 R^2 \, dR \quad R^3$

$$= \frac{r}{r^2} \int_0^r \int_0^{2\pi} X_1 R\cos\theta + X_2 R\sin\theta + X_3\sqrt{r^2-R^2} \, d\theta \, R\,dR$$

$| \, | \le \frac{r^4}{r^2} + \frac{1}{r}\int_0^r |X_3 \sqrt{r^2-R^2}| \, dR$ remains bounded

Gauss' third formula $\quad (dx = d(x_1 x_2 x_3))$

Try (1) We apply Gauss' formula (P') to f and $\frac{1}{|x|}$ and get

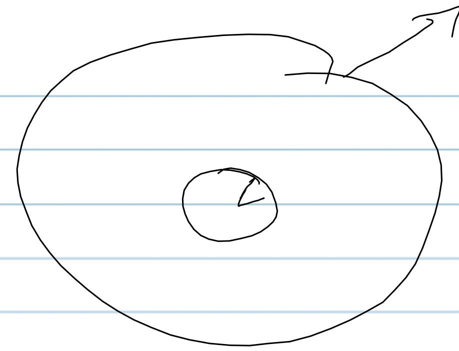
$$\int_V \Delta(f)\cdot\frac{1}{|x|} - f\,\Delta\left(\frac{1}{|x|}\right)\,dx$$

$$= \int_{\partial V}\left(\frac{\nabla f}{|x|} - f\,\underbrace{\nabla\left(\frac{1}{|x|}\right)}_{= -\frac{x}{|x|^3}}\right)\cdot n\,d\sigma(x)$$

$$= \int_{\partial V}\left(\frac{\nabla f}{|x|} + f\,\frac{x}{|x|^3}\right)\cdot n\,d\sigma(x)$$

This formula is incorrect ! because $\left.\frac{1}{|x|}\right|_{x=0} = \infty$

Try2



$$\int_{V-B(0,r)} \Delta(f)\cdot\frac{1}{|x|} - f\,\overbrace{\Delta\left(\frac{1}{|x|}\right)}^{=0}\,dx$$

$$= \int_{\partial V - B(0,r)} \left(\frac{\nabla f}{|x|} + f\frac{x}{|x|^3}\right)\cdot n\,d\sigma(x)$$

$$= \int_{\partial V}\left(\frac{\nabla f}{|x|} + f\frac{x}{|x|^3}\right)\cdot n\,d\sigma(x)$$

$$\cancel{=} \int_{\partial B(0,r)}\left(\frac{\nabla f}{|x|} + f\,\nabla\left(\frac{1}{|x|}\right)\right)\cdot n\,d\sigma(x)$$

$$\lim_{r\to 0}\int_{\partial B(0,r)} f\,\nabla\left(\frac{1}{|x|}\right)\cdot n\,d\sigma(x) = -4\pi f(0)$$

$$\lim_{r\to 0}\int_{\partial B(0,r)} \frac{\nabla f}{|x|}\cdot n\,d\sigma(x) = 0$$

Hence we get

$$\boxed{4\pi f(0) = -\int_V \Delta(f)\,\frac{dx}{|x|} + \int_{\partial V}\left(\frac{\nabla f}{|x|} + f\frac{x}{|x|^3}\right)\cdot n\,d\sigma(x)}$$

The general formula is

$$\boxed{4\pi f(r) = -\int_V \frac{\Delta(f)}{|x-r|}\,dx + \oint_{\partial V} \left(\frac{\nabla f}{|x-r|} - f\,\nabla\left(\frac{1}{|x-r|}\right)\right)\cdot n\,d\sigma}$$

Gauss 3

Philosophy recover $f(r)$ from $\Delta(f)$ and action on boundary

- Used as finding the solution of a boundary problem