

1. Practice problems-Solutions

Questions: True or false? If false give counterexample

(1) Linear independent vectors are orthogonal. no, Example

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(2) Every matrix $n \times n$ matrix a can be written as

$$a = LB$$

where B comes from a basis transform and L is lower triangular. Answer yes

(3) Let $v_1, \dots, v_m \in \mathbb{R}^k$. The Gram-Schmidt procedure gives m orthogonal vectors b_1, \dots, b_m . Answer: not necessarily, only if the v_j 's are linear independent to begin with.

Problems:

Let $C = \{(x, y) : x^2 + 2y^2 = 1\}$. Find curvature and frame at $(1/\sqrt{2}, 1/2)$.

Solution: We define

$$f(t) = \begin{pmatrix} \cos(t) \\ \frac{\sin(t)}{\sqrt{2}} \end{pmatrix}.$$

Obviously f is injective and $f[0, 2\pi] = C$. For $t = \frac{\pi}{4}$ we find $f(t) = (1/\sqrt{2}, 1/2)$. Moreover,

$$\begin{aligned} f'(t) &= \begin{pmatrix} -\sin(t) \\ \frac{\cos(t)}{\sqrt{2}} \end{pmatrix} =_{t=\pi/4} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \\ f''(t) &= \begin{pmatrix} -\cos(t) \\ -\frac{\sin(t)}{\sqrt{2}} \end{pmatrix} =_{t=\pi/4} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}, \\ (f'(t), f'(t)) &= \sin^2(t) + \frac{\cos^2(t)}{2} =_{t=\pi/4} \frac{3}{4}, \\ (f''(t), f'(t)) &= \cos(t)\sin(t) - \frac{\sin(t)\cos(t)}{2} =_{t=\pi/4} \frac{1}{4}. \end{aligned}$$

At our point this gives

$$F'(s) = \sqrt{4/3} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix},$$

and

$$|f'(t)|^2 F''(s) = f''(t) - \frac{(f'(t), f''(t))}{(f'(t), f'(t))} f'(t)$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix} - \frac{\frac{1}{2} - \frac{1}{4}}{\frac{1}{2} + \frac{1}{4}} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3\sqrt{2}} \\ -\frac{2}{3} \end{pmatrix}.$$

This gives

$$F''(s) = \begin{pmatrix} -\frac{8}{9\sqrt{2}} \\ -\frac{8}{9} \end{pmatrix}$$

and

$$\kappa(p) = \frac{8}{9} \sqrt{1 + \frac{1}{2}} = \frac{4}{9} \sqrt{5}.$$

The frame is given by $(F'(s), \frac{F''(s)}{\kappa(p)})$. ■

Find area and angle for $v_1(x) = \cos(x)$ and $v_2(x) = \sin(2x)$ in $C[-\pi, \pi]$ with

$$(f, g) = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Solution: The angle is given by

$$(v_1, v_2) = \cos(\theta)|v_1||v_2|.$$

Note that

$$(v_1, v_1) = \int_{-\pi}^{\pi} \cos^2(x)dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos^2(x) + \sin^2(x)dx = \frac{1}{\pi}.$$

Moreover, with $u = 2x$, $du = 2dx$

$$(v_2, v_2) = \int_{-\pi}^{\pi} \sin^2(2x)dx = \frac{1}{2} \int_{-2\pi}^{2\pi} \sin^2(u)du = \int_0^{2\pi} \sin^2(u)du = \frac{1}{\pi}.$$

Then with $u = -x$ we get from $\cos(x) = \cos(-x)$ and $\sin(-x) = -\sin(x)$ we find

$$(v_1, v_2) = \int_{-\pi}^{\pi} \sin(2x) \cos(x)dx = \int_{-\pi}^{\pi} \sin(-2u) \cos(-u)du = - \int_{-\pi}^{\pi} \sin(2u) \cos(u)du.$$

Thus we must have $(v_1, v_2) = 0$. Hence the angle is 90° and area is $|v_1||v_2| = \frac{1}{\pi}$. ■

Given the two lines

$$L_1 = \{(x, y, z) : x + y + z = 1, x + 2y + z = 1\}$$

and

$$L_2 = \{(x, y, z) : x + 2y + 3z = 1, 3x + y + 2z = 1\}.$$

Is there an intersection?

Solution: An intersection means that the four equations

$$\begin{aligned} x + y + z &= 1, \\ x + 2y + z &= 1, \\ x + 2y + 3z &= 1, \end{aligned}$$

$$3x + y + 2z = 1.$$

have common solution. You can check this by hand: Subtracting the second from the first equation we find $y = 0$. Indeed, all points with $y = 0$ and $x + z = 1$ solve the first two equations.

Multiply the third equation by 3 and subtract the fourth equation for $y = 0$ gives

$$9z - 2z = 2.$$

This means $z = \frac{2}{7}$, $x = \frac{5}{7}$, $y = 0$ and

$$\frac{5}{7} + \frac{6}{7} = \frac{11}{7} \neq 1.$$

So there is no solution.

You may also consider the subspace

$$V = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ 2 \end{pmatrix}\right\}.$$

Then we have a procedure to calculate the smallest distance from the point $(1, 1, 1, 1)$ to the subspace V . Indeed, we first have to find an orthonormal basis. Let us do this for

$$V = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}\right\}.$$

Then we have

$$b_1 = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$$

and

$$w_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -5 \end{pmatrix} - \frac{-16}{12} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} +\frac{1}{3} \\ \frac{4}{3} \\ \frac{4}{3} \\ -\frac{41}{9} \end{pmatrix}.$$

And then

$$\begin{aligned}
 w_3 &= \begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix} - \frac{\frac{8}{3} + \frac{41}{9}}{\frac{1}{9} + \frac{16}{9} + \frac{16}{9} + \frac{41^2}{81}} \begin{pmatrix} +\frac{1}{3} \\ \frac{4}{3} \\ \frac{4}{3} \\ -\frac{41}{9} \end{pmatrix} - \frac{-1}{12} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix} - \frac{9}{41} \frac{65}{42} \begin{pmatrix} +\frac{1}{3} \\ \frac{4}{3} \\ \frac{4}{3} \\ -\frac{41}{9} \end{pmatrix} + \begin{pmatrix} \frac{1}{12} \\ \frac{1}{12} \\ \frac{1}{12} \\ \frac{1}{36} \end{pmatrix} = \begin{pmatrix} -\frac{3 \times 65}{41 \times 42} + \frac{1}{12} \\ -\frac{12 \times 65}{41 \times 42} + \frac{1}{12} \\ 2 - \frac{12 \times 65}{41 \times 42} + \frac{1}{12} \\ -1 + \frac{65}{42} + \frac{1}{36} \end{pmatrix}
 \end{aligned}$$

Then the distance is given by the length $|w_4|$ of the vector w_4

$$v - \frac{(w_1, v)}{(w_1, w_1)} w_1 - \frac{(w_2, v)}{(w_2, w_2)} w_2 - \frac{(w_3, v)}{(w_3, w_3)} w_3$$

where $v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

■

What initial velocity is needed if you start a projectile at $(0, 0, 0)$ and want to shoot it to $(10, 10, 10)$?

Solution: Let us assume gravitation works in the negative z axis. Then we have

$$r''(t) = -ge_3,$$

Thus

$$r'(t) = v(0) - gte_3$$

and

$$r(t) = r(0) + tv(0) - g\frac{t^2}{2}e_3.$$

Here $v(0)$ is the initial velocity at $(0, 0, 0)$. This gives

$$r(t) = \begin{pmatrix} c_1 t \\ c_2 t \\ c_3 t - \frac{gt^2}{2} \end{pmatrix}.$$

Since the point $(10, 10, 10)$ is supposed to be on the trajectory we must have $c_1 = c_2$. Then we choose $t = \frac{x}{c_1}$ and find

$$r(t) = \begin{pmatrix} x \\ x \\ \frac{c_3}{c_1}x - \frac{gx^2}{2c_1^2} \end{pmatrix}.$$

This gives

$$10 = \frac{c_3}{c_1}10 - \frac{50g}{c_1^2}$$

and hence

$$10c_1^2 - 10c_3 + 50g = 0$$

and hence

$$\left(c_1 - \frac{c_3}{2}\right)^2 = \frac{c_3^2}{4} - 5g$$

and hence

$$c_1 = \frac{c_3}{2} \pm \sqrt{\frac{c_3^2}{4} - 5g}$$

describes the possible solutions (but not all of them are because negative c_1 will not get you to 10). We also see that

$$c_3^2 \geq 50g$$

is needed to get the process going at all. Indeed, $c_3 = \sqrt{50g}$ and $c_1 = c_3/2$ is quite a workable solution. ■

Find the moving frame and curvature for

$$C = \{(x \cos(x), x \sin(x), x^2) : 0 \leq x \leq 2\pi\}$$

Make a picture.

Solution: Let

$$f(x) = \begin{pmatrix} x \cos(x) \\ x \sin(x) \\ x^2 \end{pmatrix},$$

$$f'(x) = \begin{pmatrix} \cos(x) - x \sin(x) \\ \sin(x) + x \cos(x) \\ 2x \end{pmatrix},$$

$$f''(x) = \begin{pmatrix} -2 \sin(x) - \cos(x) \\ 2 \cos(x) - x \sin(x) \\ 2 \end{pmatrix}.$$

Following our experience in class we go for $x = 0$, the other points being to much calculation.

$$f'(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$f''(0) = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix},$$

and

$$(f'(0), f'(0)) = 1 \quad (f'(0), f''(0)) = -1.$$

Therefore

$$F''(s) = f''(0) - \frac{(f'(0), f''(0))}{(f'(0), f'(0))} f'(0) = f''(0) + f'(0) = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

and curvature is $\kappa(0) = 2\sqrt{2}$. The frame at that point is given by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

■

Let $H = \{(x, y, z, w) : x + y + z + w = 5\}$. Find a parametrization.

Solution: This is a three dimensional space and hence we need one point and three directions (or four points)

$$p_1 = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$p_2 = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$p_3 = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$p_4 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

This gives

$$v_1 = p_2 - p_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$v_2 = p_3 - p_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$v_3 = p_4 - p_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Looking at the last coordinates, we see that they are linearly independent. Thus we get

$$H = \{p_1 + xv_1 + yv_2 + zv_3 : x, y, z \in \mathbb{R}\}.$$

You might want to check this by checking some points.