Math 347-Homework 1

Due date: 02/02/04 (indicate partners)

(1) (20P)Let $X = \mathbb{R}^n$ and

$$d_\infty(x,y) \;=\; \max_{i=1,\dots,n} |x_i - y_i| \quad d_1(x,y) \;=\; \sum_{i=1}^n |x_i - y_i| \,.$$

Show show that d_∞ and d_1 satisfies the axioms of a metric space.

(2) (10P) 1.1 page 5

(3) (20P) 1.8 page 5

(4) (20P)Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is Lipschitz on $\mathbb{R}$, but $g(x) = x^2$ is not.

(5) (20P) We recall that an equivalence relation $\sim$ on a set X satisfies
   (a) $x \sim x$,
   (b) $x \sim y$ if and only if $y \sim x$
   (c) $x \sim y$ and $y \sim z$ implies $x \sim z$.

We denote by $[x] = \{y \;:\; y \sim x\}$. On $X = \mathbb{N}\times\mathbb{N}$ we define $(n,m) \sim (n',m')$ if

$$n + m' \;=\; n' + m \,.$$

Show that $\sim$ is an equivalence relation. Show that

$$[(n,m)] + [(n',m')] \;=\; [(n+n', m+m')]$$

is well defined and satisfies

$$[(n,m)] + [(m,n)] \;=\; [(0,0)]$$

and

$$([(n,m)] + [(n',m')]) + [(n", m")] \;=\; [(n,m)] + ([(n',m')] + [(n", m")]) \,.$$

(Remark: The set of all equivalence classes $\{[(n,m)] \;:\; n,m \in \mathbb{N}\}$ is an abelian group, namely $\mathbb{Z}$.)

Solution: a) Let (n,m), (n',m') and (n'',m'') be pairs. Since $a+b = b+a$ holds for natural numbers, we see that

$$(n,m) \sim (n',m') \Leftrightarrow n + m' = m + n' \Leftrightarrow n' + m = m' + n \Leftrightarrow (n',m') \sim (n,m) \,.$$

The relation $(n,m) \sim (n,m)$ is true because of $n+m = m+n$. Finally, we assume $(n,m) \sim (n',m')$ and $(n',m') \sim (n", m")$. Then we have

$$n + m' \;=\; n' + m \quad \text{and} \quad n' + m" = n" + m' \,.$$

This implies

$$n + m' + m" \; = \; n' + m + m" \; = \; n' + m" + m \; = \; n" + m' + m$$

Math 347-Homework 2

Due date: 02/09/04 (indicate partners) Indicate one problem you want discussed in class! Separate 1-3 and 4-6 on different pages.

(1) (15P) p-88 13.5
(2) (21P) p-89 13.9
(3) (15P) Let (x_n) be a sequence in $\mathbb{R}^d$. Show that (x_n) converges if and only if all the coordinate sequences $(x_n(j))$, $j = 1, ..., d$ converge.
(4) (10P) Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous functions on a metric space. Show that $h : X \to \mathbb{R}$ defined by $h(x) = f(x)g(x)$ is also continuous.
(5) (15P) Use the ε-δ definition to show that $f(x, y) = x^3y^2$ is continuous at $(2, 3)$.
(6) (24P) On $\mathbb{Z} \times \mathbb{Z}$ we define the equivalence relation

$$(x, y) \sim (x', y') \quad \text{if} \quad xy' \;=\; x'y$$

(As an aside multiplication of natural numbers and then integers can be defined inductively but you are assumed to have basic knowledge on this operation). We recall that $[(x, y)] = \{(x', y') \;:\; (x, y) \sim (x', y')\}$. We denote by S

(a) Show that on $\mathbb{Z} \times \mathbb{Z}/\sim$ the operation

$$[(x, y)][(x', y')] \;=\; [(xx', yy')]$$

is well defined.

(b) Show that

$$[(x, y)] + [(x', y')] \;=\; [(xy' + yx), yy']$$

is well-defined.

(c) Show

$$a(b + c) \;=\; ab + bc$$

for elements $a, b, c \in \mathbb{Z}/\sim$.

(d) For which equivalence classes $[(x, y)]$ does there exists (x', y') with

$$[(x, y)][(x', y')] \;=\; [(1, 1)]\,.$$

Math 347-Homework 3

Due date: 02/16/04

(1) (10P) Show that every convergent sequence is a Cauchy sequence.

(2) (15P) Let (X, d) a metric space and z_1, z_2, z_3 be three points such that $z_1 \neq z_2$, $z_2 \neq z_3$ and $z_1 \neq z_3$. Let $(x_n) \subset \{z_1, z_2, z_3\}$ be a sequence such that

$$\{n \in \mathbb{N} \, : \, x_n \; = \; z_i\}$$

is infinite for all $i = 1, 2, 3$. Show that (x_n) is not a Cauchy sequence.

(3) (16P) Let (X, d) be a metric space and $S \subset X$ be a so-called discrete subset, i.e. there exists a $\delta > 0$ such that $x \neq y$, $x, y \in S$ implies

$$\delta < d(x, y) \; .$$

(a) Let (x_n) be a Cauchy sequence. Show that there exists an n_δ such that

$$x_n \; = \; x_{n_\delta + 1}$$

for all $n > n_\delta$.

(b) Show that S is complete (with respect to the metric inherited from X).

(4) (10P) Show that $[0, 1)$ is not complete with the usual metric.

(5) (15P) (Recall that a series $\sum_n a_n$ of real numbers is called Cauchy if the sequence (s_n) of partial sums given by $s_n = \sum_{k=1}^{n} a_k$ is a Cauchy sequence. Show that $\sum_n |a_n|$ Cauchy implies that $\sum_n a_n$ Cauchy.

(6) (16P)

(a) Show by induction that $(1 + a)^n \; \geq \; 1 + na$ for all $a > -1$

(b) 9.18 a) on page 54. (geometric series).

Math 347-Homework 4

Due date: 02/23/04

(1) Let G be a group and $d : G \times G \to \mathbb{R}$ a metric such that

$$d(gh_1, gh_2) \;=\; d(h_1 g, h_2 g)$$

holds for all g, h_1, h_2. On the set X of sequences with elements in G. We define

$$(g_n) \sim (h_n)$$

if $\lim_n d(g_n, h_n) = 0$. On $\tilde{G} = X/\sim$ we define

$$[(g_n)][(h_n)] \;=\; [(g_n h_n)]\,. \tag{0.1}$$

(a) (5P) Show that (0.1) is well-defined, i.e. $(g_n) \sim (g_n')$ and $(h_n) \sim (h_n')$ implies $(g_n h_n) \sim (g_n' h_n')$.

(b) (10P) Let $e_{\tilde{G}} = [(e)]$. Show that $(\tilde{G}, \cdot, e_{\tilde{G}})$ is a group.

(c) (5P) What is the connection to the lecture and the construction of $\mathbb{R}$?

(2) (15P) Let X_1 and X_2 be complete metric spaces. On the product

$$X_1 \times X_2 \;=\; \{(x_1, x_2) \;:\; x_1 \in X_1, x_2 \in X_2\}$$

we define

$$d((x_1, x_2), (y_1, y_2)) \;=\; d(x_1, y_1) + d(x_2, y_2)\,.$$

Show that $(X_1 \times X_2, d)$ is again a complete metric space.

(3) (15P) Read and prove the rational 0's Theorem 2.2 and solve Exercise 2.1 (page 12).

(4) (21P) Let $a \geq 1$. We define inductively $x_0 = 0$ and

$$x_{n+1} \;=\; x_n + \frac{1}{2a^2}(a - x_n^2)\,.$$

(a) Assume that (x_n) converges. What can you say about $x = \lim_n x_n$?

(b) We will use the fact that an increasing bounded sequence converges. Show by induction

$$x_n \geq 0 \quad \text{and} \quad x_n \leq a \quad x_{n+1} \;\geq\; x_n\,.$$

(Hint: If my calculations are correct the inequality $x_n \leq a$ and hence $(2a^2 - x_n)^2 \;\geq\; (2a^2 - a)^2 \;\geq\; a$ (why is the last inequality true?) comes in quite handy.)

(c) Given an example of a Cauchy sequence or rational numbers not converging to a rational number.

(5) (20P) Let us consider once again the q-metric $d_q(x,y) = |x-y|_q$, where $|x|_q = q^{-\max\{n \,:\, q^n | x\}}$. Show that

$$s_n = \sum_{k=1}^{n} q^k$$

is Cauchy and does not have a limit in $(\mathbb{Z}, d_q)$.

Math 347-Homework 5

Due date: 02/29/04

(1) (10P) No 4.8 page 26
(2) (15) No 4.14 a) page 26
(3) (10P) No 7.4 page 36
(4) (10P) No 7.5 page 36 (with proofs)
(5) (15P) Let X and Y be metric spaces such that Y is complete and $u : X \to Y$ is a continuous bijective map with a continuous inverse $u^{-1} : u(X) \to X$. Let $A \subset X$. Show that $x \in \bar{A}$ if and only if $x \in X$ and $u(x) \in \overline{u(A)}$.
(6) (15P)
  (a) Let $0 \leq x$ and $0 \leq b_1, b_2 \leq x$ be a solutions to $b_1^2 = x = b_2^2$. Show that $b_1 = b_2$.
  (b) Let $A = \{x \in \mathbb{Q} : 0 \leq x, x^2 \leq 3\}$. Show that $\sqrt{3} = \sup A$.
  (c) Show that $\{x : x \in \mathbb{Q}, 0 \leq x, x^2 < 3\}$ is a closed subset of $\mathbb{Q}$.

Math 347-Homework 5

Due date: 04/12/04

(1) (15) Show that a polynomial of odd degree has at least one real root (= solution to $p(x) = 0$).

(2) (15) Show that there exists $x \in (0, \frac{\pi}{2})$ with $\cos(x) = x$.

(3) (15) Let f and g be diff'le on (a, b) and $f'(x) = g'(x)$ for all $a < x < b$. Show that there is a constant such that $f(x) = g(x) + C$.

(4) Let be differentiable on an open interval I and $a < b \in I$. We assume that $f'(a) < 0 < f'(b)$

   (a) (20P) Use the definition of differentiability to show that there are $a < c < d < b$ such that

   $$f(a) > f(c) \quad \text{and} \quad f(d) < f(b) .$$

   (This holds for all points c and d close enough to a, b, respectively.)

   (b) (10P) Show that the minimum on $[a, b]$ of f occurs at an point $x_0 \in (a, b)$.

   (c) (10P) Show that there exists $x_0 \in (a, b)$ with $f'(x_0) = 0$.

   (d) (15P) Show Darboux's theorem: If f is differentiable on I and $f'(a) < L < f'(b)$, then there exists $x_0 \in (a, b)$ with $f'(x_0) = L$. (Note that we don't assume f' to be continuous here!).

Math 347-Homework 9

Please separate 1-2 and 3-4

Due date: 04/26/04

1. p=222 no 29.16
2. p=222 no 29.18
3. Calculate $\sum_n \frac{n^2+n}{2^n}$.
4. Let $F(0) = F(1) = 1$ and

$$F(n+2) = F(n) + F(n+1)\ .$$

(a) Show that $F(n) \leq 2^n$ and therefore the radius R of convergence of

$$f(x) \ = \ \sum_n F(n)x^n$$

is strictly positive.

(b) Show that $f(x) = \sum_n (x+x^2)^n$ for $x < \frac{1}{2}$ by considering $f(x)(1-x-x^2)$.

(c) Compute $\lim_n F(n+1)/F(n)$ and find the radius of convergence. (Hint: Let $r_n = \frac{F(n+1)}{F(n)}$ and show by induction that $r_{n+1} - r_n = -(r_n - r_{n-1})/r_n r_{n-1}$ and that $r_n \geq 3/2$. Deduce that the limit exists and satisfies a quadratic equation. (Why do we get the radius of convergence?).

Math 347-Last Homework

(1) 34.11 p=268

(2) 34.12 p=268

(3) Show the Mean Value Theorem for integrals: If f is continuous on $[a, b]$, there there exists $c \in (a, b)$ such that

$$f(c) \;=\; \frac{1}{b-a}\int_a^b f$$

(Extra credit: show this fails for Riemann integrable not continuous f.)

(4) Show $\ln(1+x) \;=\; \sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1}$ for $|x| < 1$. Justify your steps.