

## Project

**Names:** Short term:

- (1) Show that  $(q^n)$  is a Cauchy sequence in  $(\mathbb{Z}, d_q)$ .
- (2) Show that  $x_n = (\cos(\frac{1}{n}), \sin(\frac{1}{n}))$  is not Cauchy with respect to the Chicago suburb metric.

Long term: Find the ‘completion’ of  $(\mathbb{Z}, d_q)$ . Some step towards this goal.

- (1) Let  $x \in \mathbb{Z}$ . Calculate

$$\lim_n d_q(q^n, x)$$

- (2) Let  $a \in \mathbb{Z}$  and define  $(y_n(a))$  by  $y_n = q^n + a$ . Show that  $(y_n(a))$  is Cauchy and

$$\lim_n d(y_n(a), y_n(b)) > 0$$

if  $a \neq b$ .

## Project 3

We consider the space

$$X = \prod_k \mathbb{Z}_{q^k} = \{(x_k) : x_k \in \mathbb{Z}, 0 \leq x_k < q^k\}.$$

The metric is given by

$$d((x_k), (y_k)) = \sum_{k=1}^{\infty} q^{-2k} |x_k - y_k|.$$

Note that  $d((x_k), (y_k)) \leq \frac{q}{1-1/q}$ . In the following you can use the fact that  $X$  is a complete metric space.

- (1) We define  $\mathbb{Z}_{q^\infty}$  as the subset of elements  $(x_k)$  such that in  $X$

$$x_{k+1} \cong x_k \pmod{q^k}.$$

Show that  $\mathbb{Z}_{q^\infty}$  is closed and complete.

- (2) Let  $(x_k) \in \mathbb{Z}_{q^\infty}$  and  $\varepsilon > 0$ . Show that there exists an  $x \in \mathbb{Z}$  such that the sequence of remainders

$$x = aq^k + r_k(x) \quad 0 \leq r_k(x) < q^k$$

satisfies

$$d((x_k), (r_k(x))) < \varepsilon.$$

(3) Show that map  $r : \mathbb{Z}_q \rightarrow \mathbb{Z}_{q^\infty}$  satisfies

$$(d_q(x, y))^2 \leq q^2 d(r(x), r(y))$$

and

$$d(r(x), r(y)) \leq \frac{1}{1 - \frac{1}{q}} d_q(x, y).$$

(4) (extra credit) Conclude that  $\mathbb{Z}_{q^\infty}$  is a completion of  $\mathbb{Z}_q = (\mathbb{Z}, d_q)$ .

### Project 3

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Note that  $d((x_k), (y_k)) \leq \frac{q}{1-1/q}$ . In the following you can use the fact that  $X$  is a complete metric space.

(1) We define  $\mathbb{Z}_{q^\infty}$  as the subset of elements  $(x_k)$  such that in  $X$

$$x_{k+1} \cong x_k \pmod{q^k}$$

for all  $k \in \mathbb{N}$ . Show that  $\mathbb{Z}_{q^\infty}$  is closed and complete.

**Solution:** Let  $x = (x_k) \in X$  such that  $x \notin \mathbb{Z}_{q^\infty}$ . This means there exists a  $k$  such that

$$x_{k+1} \not\cong x_k \pmod{q^k}.$$

We define  $\delta = q^{-2(k+1)}$ . Let  $y = (y_k)$  such that

$$d(x, y) < \delta.$$

This implies that

$$|x_k - y_k| \leq q^{2k} d(x, y) < q^{2k} q^{-2(k+1)} < 1$$

and

$$|x_{k+1} - y_{k+1}| \leq q^{2(k+1)} d(x, y) < q^{2(k+1)} q^{-2(k+1)} = 1.$$

Therefore  $x_k = y_k$  and  $x_{k+1} = y_{k+1}$  and in particular  $y_{k+1}$  is not congruent  $y_k$  modulo  $q^k$ . ■

- (2) Let  $(x_k) \in \mathbb{Z}_{q^\infty}$  and  $\varepsilon > 0$ . Show that there exists an  $x \in \mathbb{Z}$  such that the sequence of remainders

$$x = aq^k + r_k(x) \quad 0 \leq r_k(x) < q^k$$

satisfies

$$d((x_k), (r_k(x))) < \varepsilon .$$

**Solution:** Let  $k$  be such that  $q^{k+1} > \frac{2}{\varepsilon(1-1/q)}$ . We define  $y = x_k$ . Then we have

$$x_{k-1} \cong x_k \cong y \pmod{q^{k-1}} .$$

In particular,  $r_{k-1}(y) = x_{k-1}$ . Inductively, we get  $r_j(y) = x_j$  for  $j = 1, \dots, k$ . Therefore, we get

$$\begin{aligned} d((x_k), r(y)) &= \sum_{j>k} q^{-2j} |x_j - y| \leq \sum_{j>k} q^{-2j} 2q^j \leq 2q^{-(k+1)} \sum_{j=0}^{\infty} q^{-j} \\ &= \frac{2q^{-(k+1)}}{1-1/q} < \varepsilon . \end{aligned}$$

The assertion is proved. ■

- (3) Show that map  $r : \mathbb{Z}_q \rightarrow \mathbb{Z}_{q^\infty}$  satisfies

$$(d_q(x, y))^2 \leq q^2 d(r(x), r(y))$$

and

$$d(r(x), r(y)) \leq \frac{1}{1-\frac{1}{q}} d_q(x, y) .$$

**Solution:** Let  $x, y \in \mathbb{Z}$  and define  $z = (y - x)$ . Let  $m$  be maximal such that  $q^m$  divides  $z$ . Let  $j < m$ . We write

$$x = aq^j + r_j(x) \quad \text{and} \quad y = bq^j + r_j(y) .$$

Then

$$cq^m = x - y = aq^j - bq^j + r_j(x) - r_j(y)$$

This implies  $r_j(x) - r_j(y) = 0$  and hence  $r_j(x) = r_j(y)$ . Therefore, we get

$$\begin{aligned} d(r(x), r(y)) &\leq \sum_{j=m}^{\infty} q^{-2j} |r_j(x) - r_j(y)| \\ &< \sum_{j=m}^{\infty} q^{-2j} q^j \leq \frac{q^{-m}}{1-1/q} = \frac{d_q(x, y)}{1-1/q} . \end{aligned}$$

This proves the second inequality. For the first inequality, we assume  $d(r(x), r(y)) = a$ . Then, we have for all  $k \in \mathbb{N}$ .

$$|r_k(x) - r_k(y)| \leq q^{2k} d(x, y) \leq q^{2k} a .$$

Choose  $k$  maximal with  $q^{2k} a < 1$ . This means

$$q^{2k} < \frac{1}{a} \leq q^{2(k+1)} = q^2 q^{2k} .$$

This implies

$$(q^{-k})^2 \leq q^2 a .$$

If  $a \geq 1$ , we have  $d_q(x, y) \leq 1$  and hence

$$d_q(x, y)^2 \leq a .$$

For  $a \leq 1$  we find such a  $k$  and hence  $r_k(x) = r_k(y)$ . This implies  $q^k |x - y|$ . In particular,

$$d_q(x, y)^2 \leq q^{-2k} \leq q^2 a = q^2 d(r(x), r(y)) .$$

The assertion is proved. ■

(4) (extra credit) Conclude that  $\mathbb{Z}_{q^\infty}$  is a completion of  $\mathbb{Z}_q = (\mathbb{Z}, d_q)$ .

**Solution:** From the last assertion we deduce that  $(x^j)$  is a Cauchy sequence in  $(\mathbb{Z}, d_q)$  if and only if  $(r(x^j))$  is Cauchy in  $\mathbb{Z}_{q^\infty}$ . Since a closed subset of a complete metric space is complete we see that  $r(x^j)$  converges to an element  $x \in \mathbb{Z}_{q^\infty}$ . Since  $r(\mathbb{Z})$  is dense in  $\mathbb{Z}_{q^\infty}$ , we see that  $\mathbb{Z}_{q^\infty}$  is a complete metric space which contains a dense copy of  $(\mathbb{Z}, d_q)$ . This is the definition for completion.

## 1. Project 4

**Due: Friday, February 26**

Let  $F$  be a complete totally ordered field. Show that  $F$  is archimedean.

### Project 6

Due Friday March 12

- (1) Find all the statements in the lecture where we constructed a converging sequence.
- (2) Read again the solutions to the project on the completion of  $\mathbb{Z}$  with the q-adic metric
- (3) Redo the exam (students with A's) are excused.

Exam:

- (1) (25P) Let  $A \subset \mathbb{R}$  be a closed bounded set. Show that

$$\sup A \in A.$$

Extra credit: Show that  $\sup A \in A$  if and only if  $\{b : A \leq b\}$  is open.

- (2) (30P) We consider our favorite metric space  $(\mathbb{Z}, d_q)$  with the  $q$ -adic metric. For  $x \in \mathbb{Z}$ , we recall  $r_k(x)$  to be the remainder with respect to  $q^k$ , i.e.

$$x = a_k q^k + r_k(x), \quad 0 \leq r_k(x) < q^k$$

for some  $a_k \in \mathbb{Z}$ . Show that the map  $u : \mathbb{Z}_q \rightarrow \mathbb{R}$  (usual metric for  $\mathbb{R}$ ) defined by

$$u(x) = \sum_{k=1}^{\infty} r_k(x) q^{-2k}$$

is Lipschitz.

- (3) Let  $F$  be an totally ordered archimedean field.
- (a) (10P) Let  $a \in F$  and  $a > 0$ . Show that there is an integer such that  $a < n < a + 2$ .
  - (b) (15P) Let  $0 < a < b \in F$ . Show that there exists an integer such that  $a < \frac{n}{k} < b$ . (Hint: First choose  $k$  such that  $a + \frac{2}{k} < b$  and then try to apply (a) for  $a' = \frac{k}{2}$  something).
  - (c) (10P) Show that  $\mathbb{Q}$  is dense in  $F$ .
  - (d) (Extra, extra credit) Show that two totally ordered complete fields are isomorphic (Hint  $\mathbb{Q}$  is dense in both of them).