1. Compact sets

Definition 1.1. *A collection (V_i) of open sets is called an open cover for a set K if for every $x \in K$ there exists $i \in I$ such that*

$$x \in V_i \, .$$

Let (V_i) be an open cover of a set. A collection $(B_j)_{j \in j}$ is called a subcover, if for every $x \in K$ there exists $i \in J$ and $i \in I$ such that

$$x \in B_j \subset V_i \, .$$

Remark 1.2. *In metric space every open cover has a subcover with balls (centered at elements in K).*

Definition 1.3. *A subset K of a metric space is compact if every open cover has a finite subcover.*

Theorem 1.4. $[0,1]$ *is compact.*

Theorem 1.5. *Let K be a compact subset of a metric space and $u : K \to \mathbb{R}$, then there exists a $k_{max} \in K$ such that*

$$u(k_{max}) \;=\; \sup\{u(k) \;:\; k \in K\} \, .$$

Definition 1.6. *A subset $K \subset X$ is called totally bounded if for every $\varepsilon > 0$ there exists $x_1, ..., x_m \in X$ such that*

$$K \subset B(x_1, \varepsilon) \cup \cdots \cup B(x_m, \varepsilon) \, .$$

Lemma 1.7. *A compact set is totally bounded.*

Lemma 1.8. *(Grid method) If K is totally bounded and $(x_n) \subset K$ is a sequence, then there exists a Cauchy subsequence (x_{n_k}).*

Lemma 1.9. *A compact set is complete.*

Lemma 1.10. *If X has a countable dense subset and $K \subset X$, then every open cover of K has a countable subcover.*

Theorem 1.11. *Let (X, d) be metric space (with a dense countable set D) and $K \subset X$. TFAE*

(1) *K is compact.*
(2) *K is complete and totally bounded.*
(3) *Every sequence has a convergent subsequence.*