1. Continuous functions and sequences of continuous functions

DEFINITION 1.1. *Let (X, d) be a metric space. A sequence (f_n) of functions is called equi-continuous if for every $x \in X$, $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

for all $y \in X$.

LEMMA 1.2. *Let (f_n) be a sequence of equi-continuous functions such that*

$$f(x) \; = \; \lim_n f_n(x)$$

exists for all $x \in X$. Then the limit function f is continuous.

Remark: This means that $C_b(X, \mathbb{R})$ is complete.

DEFINITION 1.3. *(f_n) converges uniformly to f if*

$$\lim_n \sup_{x \in X} |f_n(x) - f(x)| \; = \; 0 \, .$$

Project: (Dini's theorem)

LEMMA 1.4. *Let (a_n) be a sequence and $C > 0$ such that $|a_n| \; \leq \; C^n$ for all $n \geq n_0$. Then the sequence of partial sums (f_n) defined*

$$f_n(x) \; = \; \sum_{k=1}^{\infty} a_k x^k$$

is equi-continuous on $(-C, C)$.

COROLLARY 1.5. *Under the assumptions above*

$$f(x) \; = \; \sum_{k=1}^{\infty} a_k x^k$$

is well-defined and a continuous function on $(-C, C)$.

Examples: $\sum_n (-1)^n x_n$. $e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!}$. $\sin(x)$, $\cos(x)$.

PROPOSITION 1.6. $\lim(1 + \frac{1}{n})^n \; = \; e^1$.

Project: Mean-value theorem for continuous functions.

PROPOSITION 1.7. *Let $f : [a, b] \to [c, d]$ be a monotone continuous function, then the inverse function is continuous.*

Application: $e^{x+y} \; = \; e^x e^y$, $\ln(xy) \; = \; \ln(x) + \ln(y)$.

Application: $f(x) = x^n$ yields $x^{\frac{1}{n}} = f^{-1}(x)$.

DEFINITION 1.8. *If $a \in \mathbb{R}$ and $b > 0$ we define*

$$b^a \; = \; e^{a \ln(b)} \, .$$

Properties: $(b_1b_2)^a = b_1^a b_2^a$. $b^{a_1+a_2} = b^{a_1}b^{a_2}$. $(b^{a_1})^{a_2} = b^{a_1a_2}$ In particular,

$$(b^{\frac{1}{n}})^n = b\,.$$