

1. Differentiation

DEFINITION 1.1. Let O be open and $f : O \rightarrow \mathbb{R}$ be a function. f is called differentiable if

$$\lim_{t \rightarrow 0} \frac{f(x+t) - f(x)}{t}$$

exists. (What does this mean?) In this case f is called differentiable at x .

REMARK 1.2. $\lim_{x \rightarrow x_0} g(x) = a$ iff $\lim_n x_n \rightarrow x_0, g(x_n) = a$ for every sequence (x_n) converging to x_0 .

PROPOSITION 1.3. (1) If f differentiable at x implies f continuous at x .

(2) f, g differentiable at x and $\lambda \in \mathbb{R}$, then

$$(f + \lambda g)'(x) = f'(x) + \lambda g'(x), (fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

If moreover $f(x) \neq 0$, then

$$(1/f)'(x) = -f'(x)/f(x).$$

PROPOSITION 1.4. Let f be a monotone function such that $f'(x) \neq 0$. Then f^{-1} is differentiable at $f(x)$ and

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

PROPOSITION 1.5. (Rolle). Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous such that $f|_{(a,b)}$ is differentiable. If $f(a) = f(b)$, then there exists $a < x < b$ with $f'(x) = 0$.

COROLLARY 1.6. (Mean-value) Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous such that $f|_{(a,b)}$ is differentiable. Then there exists $a < x < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(x).$$

Examples: Concrete error estimates from the mean-value theorem. $\sin(x) \leq x$.

THEOREM 1.7. Let (f_n) be a sequence of functions defined on (a, b) such that $f(x) = \lim_n f_n(x)$ exists and (f'_n) converges uniformly to a function g . Then

$$f'(x) = g(x).$$

COROLLARY 1.8. Let $R = (\limsup_n |a_n|^{\frac{1}{n}})^{-1}$. Then

$$\left(\sum_k a_k x^k\right)' = \sum_k k a_k x^{k-1}$$

on $(-R, R)$.

COROLLARY 1.9. $(e^x)' = e^x$. $\ln'(x) = \frac{1}{x}$.