1. Metric Spaces

We want to generalize the concepts of convergence of sequences of real numbers to higher dimensions and similar objects. We will do this considering metric spaces.

DEFINITION 1.1. *A metric space is given by a set X and a distance function $d : X \times X \to \mathbb{R}$ such that*

i) *(Positivity) For all $x, y \in X$*

$$0 \le d(x, y) \, .$$

ii) *(Non-degenerated) For all $x, y \in X$*

$$0 \; = \; d(x, y) \Rightarrow x \; = \; y \, .$$

iii) *(Symmetry) For all $x, y \in X$*

$$d(x, y) \; = \; d(y, x)$$

iv) *(Triangle inequality) For all $x, y, z \in X$*

$$d(x, y) \; \le \; d(x, z) + d(z, y) \, .$$

Examples:

i) $X = \mathbb{R}$, $d(x, y) = |x - y|$.

ii) $X = \mathbb{R} \times \mathbb{R}$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$d_1(x, y) \; = \; |x_1 - y_1| + |x_2 - y_2| \, .$$

iii) $X = \mathbb{R} \times \mathbb{R}$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$d_2(x, y) \; = \; \left(|x_1 - y_1|^2 + |x_2 - y_2|^2\right)^{\frac{1}{2}} \; .$$

iv) Let $X = \{p_1, p_2, p_3\}$ and

$$\begin{aligned} d(p_1, p_2) &= d(p_2, p_1) = 1 \, , \\ d(p_1, p_3) &= d(p_3, p_1) = 2 \, , \\ d(p_2, p_3) &= d(p_3, p_2) = 3 \, . \end{aligned}$$

Can you find a triangle (p_1, p_2, p_3) in the plane with these distances?

v) Let $X = \{p_1, p_2, p_3\}$ and

$$\begin{aligned} d(p_1, p_2) &= d(p_2, p_1) = 1 \, , \\ d(p_1, p_3) &= d(p_3, p_1) = 2 \, , \\ d(p_2, p_3) &= d(p_3, p_2) = 4 \, . \end{aligned}$$

Can you find a triangle (p_1, p_2, p_3) in the plane with these distances? No, this is not an example of a metric space.

vi) The French railway metric on $X = \mathbb{R}^2$ is defined as follows: Let $x_0 = (0,0)$ be the origin, then

$$d_{SNCF}(x,y) \;=\; \begin{cases} d_2(x,y) & \text{if there exists a } t \in \mathbb{R} \text{ such that } x_1 = ty_1 \\ & \text{and } x_2 = ty_2 \\ & \\ d_2(x,x_0) + d_2(x_0,y) & \text{else} \end{cases} .$$

It is by no means trivial to show that d_2 satisfies the triangle inequality. In the following we write $0 = (0, ..., 0)$ for the origin in $\mathbb{R}^n$.

Lemma 1.2. *Let $x, y \in \mathbb{R}^n$, then*

$$|\sum_{i=1}^n x_i y_i| \;\le\; \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n |y_i|^2\right)^{\frac{1}{2}}$$

Proof: We can assume $d_2(y,0) > 0$. Let $t \in \mathbb{R}$ be arbitrary, then

$$0 \;\le\; \sum_{i=1}^n (x_i + ty_i)^2 \;=\; \sum_{i=1}^n x_i^2 + t^2 \sum_{i=1}^n y_i^2 + 2t \sum_{i=1}^n x_i y_i \,.$$

Let us assume $\sum_{i=1}^n x_i y_i > 0$ first and put $t = -\frac{d_2(x,0)}{d_2(y,0)}$ then we get

$$|t| \left|\sum_{i=1}^n x_i y_i\right| \;\le\; \frac{1}{2} d_2(x,0)^2 + \frac{1}{2} t^2 d_2(y,0)^2 \;=\; d_2(x,0)^2 \,.$$

Thus

$$\left|\sum_{i=1}^n x_i y_i\right| \;\le\; d_2(x,0)^2 \frac{d_2(y,0)}{d_2(x,0)} \;=\; d_2(x,0) d_2(y,0) \,.$$

The case $\sum_{i=1}^n x_i y_i \le 0$ is similar. ■

Lemma 1.3. *On $\mathbb{R}^n$ the metric*

$$d_2(x,y) \;=\; \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{\frac{1}{2}}$$

satisfies the triangle inequality.

Proof: Let $x, y, z \in \mathbb{R}^n$. Then we deduce from Lemma 1.2

$$d(x,y)^2 \;=\; \sum_{i=1}^n |x_i - y_i|^2 \;=\; \sum_{i=1}^n |(x_i - z_i) - (y_i - z_i)|^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} |(x_i - z_i)|^2 - 2\sum_{i=1}^{n}(x_i - z_i)(y_i - z_i) + \sum_{i=1}^{n} |y_i - z_i|^2 \\
&\le d(x,z)^2 + 2d(x,y)d(y,z) + d(y,z) \\
&= (d(x,z) + d(y,z))^2 \ .
\end{aligned}
$$

Hence,

$$d(x,y) \ \le \ d(x,z) + d(y,z) \ .$$

and the assertion is proved. ■

Exercise: Show that the railroad metric satisfies the triangle inequality.

1.1. Complete metric spaces. We will say that a sequence in a metric space is a Cauchy sequence of for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon$$

for all $n, m > n_0$. We will say that a sequence (x_n) in metric space is convergent if there exists $x \in X$ such that for every $\varepsilon > 0$ there exists an n_0 such that for $n > n_0$

$$d(x_n, x) < \varepsilon$$

In this case we write

$$\lim_n x_n \; = \; x \; .$$

DEFINITION 1.4. *A metric space (X, d) is called comlete, if every Cauchy sequence converges.*

PROPOSITION 1.5. *The space $(\mathbb{R}^2, d_1)$ is complete.*

Proof: Let x_n be a Cauchy sequence in $(\mathbb{R}^2, d_1)$. Then $x_n = (x_n(1), x_n(2))$ is a sequence of pairs.
Claim: The sequences $(x_n(1))_{n\in\mathbb{N}}$ and $(x_n(2))_{n\in\mathbb{N}}$ are Cauchy sequences.
Indeed, let $\varepsilon > 0$, then there exists an n_0 such that

$$d_1(x_n, x_m) < \varepsilon$$

for all $n, m > n_0$. In particular, we have

$$|x_n(1) - x_m(1)| \; \leq \; |x_n(1) - x_m(1)| + |x_n(2) - x_m(2)| \; \leq \; d_1(x_n, x_m) < \varepsilon$$

for all $n, m > n_0$ and

$$|x_n(2) - x_m(2)| \; \leq \; |x_n(1) - x_m(1)| + |x_n(2) - x_m(2)| \; \leq \; d_1(x_n, x_m) < \varepsilon \; .$$

Therefore, $(x_n(1))$ and $(x_n(2))$ is Cauchy.
Since $\mathbb{R}$ is complete, we can find $x(1)$ and $x(2)$ such that

$$\lim_n x_n(1) \; = \; x(1) \quad \text{and} \quad \lim_n x_n(2) \; = \; x(2) \; .$$

Claim: $\lim_n x_n \; = \; (x(1), x(2))$.
Indeed, Let $\varepsilon > 0$ and choose n_1 such that

$$|x_n(1) - x(1)| < \frac{\varepsilon}{2}$$

for all $n > n_1$. Choose n_2 such that

$$|x_n(2) - x(2)| < \frac{\varepsilon}{2}$$

for all $n > n_2$. Set $n_0 = \max\{n_1, n_2\}$, then for every $n > n_0$, we have

$$d_1(x_n, (x(1), x(2)) \;=\; |x_n(1) - x(1)| + |x_n(2) - x(2)| \;<\; \varepsilon$$

Thus

$$\lim_n x_n \;=\; x$$

and the assertion is proved. ■

Examples:

(1) Let $X = \mathbb{R} \setminus \{0\}$ and $d(x, y) = |x - y|$, them (X, d) is not complete. The sequences $(\frac{1}{n})$ is Cauchy and does not converge.

(2) Let p be a prime number. On the set of integers, we define

$$dd_p(z, w) \;=\; p^{-n} \,,$$

where $n = \max\{n \;:\; p^n \text{ divides } (z - w)\ \}$. This satisfies the triangle inequality. The sequence x_n given by $x_n = p^n$ is a non convergent Cauchy sequence.

THEOREM 1.6. *Let $n \in \mathbb{N}$. The space $(\mathbb{R}^n, d_2)$ is a complete metric space.*

Proof: Similar as in Proposition 1.5 using the following Lemma . ■

LEMMA 1.7. *Let $x, y \in \mathbb{R}^n$, then*

$$d_2(x, y) \;\le\; \sum_{i=1}^{n} |x_i - y_i| \,.$$

Proof: We proof this by induction on $n \in \mathbb{N}$. The case $n = 1$ is obvious. Assume the assertion is true for n and let $x, y \in \mathbb{R}^{n+1}$. We define the element $z = (x_1, ..., x_n, y_{n+1})$, then we deduce from the triangle inequality

$$\begin{aligned} d_2(x, y) &\le d_2(x, z) + d_2(z, y) \\ &= \left(\sum_{i=1}^{n+1} |x_i - z_i|^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n+1} |z_i - y_i|^2 \right)^{\frac{1}{2}} \\ &= |x_{n+1} - y_{n+1}| + \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}} \,. \end{aligned}$$

To apply the induction hypothesis, we define $\tilde{x} = (x_1, ..., x_n)$ and $\tilde{y} = (y_1, .., y_n)$. Then the induction hypothesis yields

$$\left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}} \;=\; d_2(\tilde{x}, \tilde{y}) \;\le\; \sum_{i=1}^{n} |x_i - y_i| \,.$$

Hence,

$$\begin{aligned} d_2(x,y) &\leq |x_{n+1} - y_{n+1}| + \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}} \\ &\leq |x_n - y_n| + \sum_{i=1}^{n} |x_i - y_i| \\ &= \sum_{i=1}^{n+1} |x_i - y_i| \, . \end{aligned}$$

The assertion is proved. ■

1.2. Continuous functions and open sets. Let (X, d) and (Y, d') be metric spaces. A map $f : X \to Y$ is called continuous if for every $x \in X$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(x,y) < \delta \quad \Longrightarrow \quad d'(f(x), f(y)) < \varepsilon \, . \tag{1.1}$$

Let us use the notation

$$B(x,\delta) \;=\; \{y \,:\, d(x,y) < \delta\} \, .$$

For a subset $A \subset X$, we also use the notation

$$f(A) \;=\; \{f(x) \,:\, x \in A\} \, .$$

Similarly, for $B \subset Y$

$$f^{-1}(B) \;=\; \{x \in X \,:\, f(x) \in B\} \, .$$

Then (1.1) means

$$f(B(x,\delta)) \;\subset B(f(x),\varepsilon) \, .$$

Or in a very non-formal way

$$\text{f maps small balls into small balls} \quad .$$

Our aim is to prove a criterion for continuity in terms of so called open sets. This criterion illustrates simultaneously the role of open sets and its interaction with continuity and has a genuinely geometric flavor. A subset O of a metric space is called <u>open</u> if

$$\forall x \in O \;: \exists \delta > 0 \;: B(x,\delta) \subset O \, .$$

For example the sets

$$O \;=\; (-1,1) \, , \; O \;=\; \mathbb{R} \, , \; O \;=\; (-1,1) \times (-2,2)$$

are open in $\mathbb{R}$, $(\mathbb{R}^2, d_2)$ respectively.

PROPOSITION 1.8. *Let (X, d), (Y, d') be metric spaces and $f : X \to Y$ be a map. f is contiuous iff $f^{-1}(O)$ is open for all open subsets $O \subset Y$.*

Proof: $\Rightarrow$. We assume that f is continuous and O is open. Let $x \in f^{-1}(O)$, i.e. $f(x) \in O$. Since O is open, there exists an $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset O$. By continuity, there exists a $\delta > 0$ such that

$$f(B(x, \delta)) \subset B(f(x), \varepsilon) \subset O .$$

Therefore

$$B(x, \delta) \subset f^{-1}(O) .$$

Since $x \in f^{-1}(O)$ was arbitrary, we deduce that $f^{-1}(O)$ is open.
$\Leftarrow$ Let $x \in X$ and $\varepsilon > 0$. Let us show that

$$B(f(x), \varepsilon)$$

is a on open subset of (Y, d'). Indeed, let $y \in B(f(x), \varepsilon)$ define $\varepsilon' = \varepsilon - d'(y, f(x))$. Let $z \in Y$ such that

$$d(z, y) < \varepsilon'$$

then

$$d(f(x), z) \;\leq\; d(f(x), y) + d(y, z) \;<\; d(f(x), y) + \varepsilon - d'(y, f(x)) \;=\; \varepsilon .$$

Thus

$$B(y, \varepsilon - d'(f(x), y)) \;\subset B(f(x), \varepsilon) .$$

By the assumption, we see that $f^{-1}(B(f(x), \varepsilon))$ is an open set. Since $x \in f^{-1}(B(f(x), \varepsilon))$, we can find a $\delta > 0$ such that

$$B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon)) .$$

Hence, for all $\tilde{x}$ with $d(x, \tilde{x}) < \delta$, we have

$$d'(f(x), f(\tilde{x})) < \varepsilon .$$

The assertion is proved. ■

REMARK 1.9. *The sets $B(x, \varepsilon)$, $x \in X$, $\varepsilon > 0$ are open.*

In the following we consider the metrics

$$d_p(x, y) \;=\; \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

and

$$d_\infty(x, y) \;=\; \max_{i=1,..,n} |x_i - y_i|$$

on $\mathbb{R}^n$.

LEMMA 1.10. *Let $1 \leq p \leq \infty$ and $x, y \in \mathbb{R}^n$, then*

$$\frac{1}{n^{\frac{1}{p}}}\, d_p(x,y) \;\leq\; d_\infty(x,y) \;\leq\; d_p(x,y)\ .$$

Proof: The last inequality is obvious. For the first one, we consider $x, y \in \mathbb{R}^n$ and $1 \leq p < \infty$, then by estimating every element in the sum against the maximum

$$d_p(x,y)^p \;=\; \sum_{i=1}^{n} |x_i - y_i|^p \;\leq\; n \max\{|x_i - y_i|^p\}\ .$$

Taking the p-th root, we deduce the assertion. ■

COROLLARY 1.11. *Let $1 \leq p, q \leq \infty$, then the identity map $id : (\mathbb{R}^n, d_p) \to (\mathbb{R}^n, d_q)$ is continuous.*

Proof: We have for all $x \in \mathbb{R}^n$ and $\varepsilon > 0$

$$B_{d_p}(x, \frac{\varepsilon}{n}) \subset B_{d_q}(x, \varepsilon)\ .$$

This easily implies the assertion. ■

COROLLARY 1.12. *The metrics d_p define the same open sets on $\mathbb{R}^n$.*

Warning: We didn't show that d_p satisfies the triangle inequality and we will not do it in this course.

PROPOSITION 1.13. *Let (X, d), (Y, d') be metric spaces and $f : X \to Y$ be a map. Then f is continuous if for every convergent sequence (x_n) in X*

$$\lim_n f(x_n) \;=\; f(\lim_n x_n)\ .$$

Proof: $\Rightarrow$ Let $x = \lim_n x_n$ and $\varepsilon > 0$, then there exists a $\delta > 0$ such that

$$d(y,x) < \delta \Rightarrow d'(f(y), f(x)) < \varepsilon\ .$$

Let $n_0 \in \mathbb{N}$ be such that

$$d(x_n, x) < \delta$$

for all $n > n_0$, then

$$d'(f(x_n), f(x)) < \varepsilon$$

for all $n > n_0$. Hence

$$\lim_n f(x_n) \;=\; f(x)\ .$$

$\Leftarrow$ Let $x \in X$ and assume in the contrary that

$$\exists \varepsilon > 0\, \forall \delta > 0 \exists y\ : d(y,x) < \delta \text{ and } d'(f(x), f(y)) \geq \varepsilon\ .$$

Applying these successively for all $\delta = \frac{1}{k}$, we find a sequence (x_k) such that

$$d(x_k, x) \; < \; \frac{1}{k} \quad \text{and} \quad d'(f(x_k), f(x)) \geq \varepsilon' \; .$$

and thus

$$\lim_k x_k \; = \; x \; .$$

By assumption, we have

$$\lim_k f(x_k) \; = \; f(x) \; .$$

Hence, there exists a k_0 such that for all $k > k_0$

$$d(f(x_k), f(x)) < \varepsilon \; .$$

a contradiction. ∎

2. Continuation of continuity

EXAMPLE 2.1. (1) Let (X, d) be a metric space and $x_0 \in X$ be a point , then $f(x) = d(x, x_0)$ is continuous. Indeed, the triangle inequality implies

$$d(d(x, x_0), d(d(y, 0)) \;=\; |d(x, x_0) - d(y, x_0)| \;\leq\; d(x, y)$$

This easily implies the assertion.

(2) On $\mathbb{R}^n$ with the standard euclidean metric $d = d_2$, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(x) = d(x, 0)x$ is continuous.

Proof: Let $x \neq 0$ a pint in $\mathbb{R}^n$ and let $\varepsilon > 0$. Choose $\delta \;=\; \min\{\sqrt{\frac{\varepsilon}{2}}, \frac{\varepsilon}{2d(x,0)}\}$, Let $d(y, x) < \delta$, then by the above inequality

$$\begin{aligned} d(f(x), f(y)) &= d(d(x,0)x, d(y,0)y) \;\leq\; d(d(x,0)x, d(x,0)y) + d(d(x,0)y, d(y,0)y) \\ &= d(x,0)d(x,y) + |d(x,0) - d(y,0)|d(y,0) \\ &\leq d(x,0)d(x,y) + d(x,y)(d(x,0) + d(x,y)) \\ &= 2d(x,0)d(x,y) + d(x,y)^2 \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon. \end{aligned}$$

Now let $x = 0$ and $\varepsilon > 0$ and define $\delta = \sqrt{\varepsilon}$, then for $d(x, y) < \delta$ we have

$$d(f(x), f(y)) \;=\; d(0, d(0,y)y) \;=\; d(0,y)^2 < \varepsilon\ .$$

The assertion is proved. ■

(3) (Exercise) The function $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (\cos(x_1), \sin(x_2), \cos(x_1))$ is continuous. Try that one for the next hour exam.

3. Closed and Compact Sets

Let (X, d) be a metric space. We will say that a subset $A \subset X$ is *closed* if $X \setminus A$ is open.

PROPOSITION 3.1. *Let (X, d) be a complete metric space and $C \subset X$ a subset. C is closed iff every Cauchy sequence in C converges to an element in C.*

Proof: Let us assume C is closed and that (x_n) is a Cauchy sequence with elements in C. Let $x = \lim_n x_n$ be te limit and assume $x \notin C$. Since $X \setminus C$ is open

$$B(x, \varepsilon) \subset X \setminus C$$

for some $\varepsilon > 0$. Then there exists an n_0 such that $d(x_n, x) < \varepsilon$ for $n > n_0$. In particular,

$$x_{n_0+1} \in B(x, \varepsilon)$$

and thus $x_{n_0+1} \notin C$, a contradiction.
Now, we assume that every Cauchy sequence with values in C converges to an element in C. If $X \setminus C$ is not open, then there exists an $x \notin C$ and no $\varepsilon > 0$ such that

$$B(x, \varepsilon) \subset X \setminus C\,.$$

I.e. for every $n \in \mathbb{N}$, we can find $x_n \in C$ such that

$$d(x, x_n) < \frac{1}{n}\,.$$

Hence, $\lim x_n = x \in C$ but $x \notin C$, contradiction. ■
The most important notion in this class is the notion of compact sets. We will say that a subset $C \subset X$ is *compact* if For every collection (O_i) of open sets such that

$$C \subset \bigcup_i O_i \;=\; \{x \in X \mid \exists_{i \in I} x \in O_i\}$$

There exists $n \in \mathbb{N}$ and $i_1, ..., i_n$ such that

$$C \subset O_{i_1} \cup \cdots \cup O_{i_n}\,.$$

In other words

Every open cover of C has a finite subcover .

Definition 3.2. *Let* $X \subset \bigcup O_i$ *be an open cover. Then we say that* (V_j) *is an open subcover if*

$$X \subset \bigcup_j V_j$$

all the V_j *are open and for every* j *there exists an* i *such that*

$$V_j \subset O_i \ .$$

It is impossible to explain the importance of 'compactness' right away. But we can say that there would be no discipline 'Analysis' without compactness. The most clarifying idea is contained in the following example.

Proposition 3.3. *The set* $[0,1] \subset \mathbb{R}$ *is compact.*

Proof: Let $[0,1] \subset \bigcup_i O_i$. For every $x \in [0,1]$ there exists an $i \in I$ such that

$$x \in O_i \ .$$

Since O_i is open, we can find $\varepsilon > 0$ such that

$$x \in B(x,\varepsilon) \subset O_i \ .$$

Using the axiom of choice, we fine a function ε_x and i_x such that

$$x \in B(x,\varepsilon_x) \subset O_{i_x} \ .$$

Let us define the relation $x \preceq y$ if $x < y$ and

$$y - x \ \leq \ e_x + e_y \ .$$

The crucial point here is to define

$$S \ = \ \{x \in [0,1] \mid \exists \, x_1, ..., x_n \ : \frac{1}{2} \preceq x_1 \preceq \cdots \preceq x_n \preceq x\} \ .$$

We claim a) $\sup S \in S$ and b) $\sup S = 1$.

Ad a): Let $y = \sup S \in [0,1]$. Then there exists an $x \in S$ with

$$y - \varepsilon_y < x \leq y \ .$$

Then obviously $x \preceq y$. Since $x \in S$, we can find

$$\frac{1}{2} \preceq x_1 \preceq \cdots \preceq x_n \preceq x \preceq y \ .$$

Thus $y \in S$.

Ad b): Assume $y = \sup S < 1$. Let $0 < \delta \ = \ \min(e_y, 1-y)$. Then

$$y + \delta - y \ = \ \delta \ \leq \ \varepsilon_y + \varepsilon_{y+\delta} \ .$$

By a), we find

$$\frac{1}{2} \preceq x_1 \preceq \cdots \preceq x_n \preceq y \preceq y + \delta$$

and thus $y + \delta \in S$. Contradiction to the definition of the supremum. Assertion a) and b) are proved.

Therefore we conclude $1 \in S$ and thus find $x_1, ..., x_n$ such that

$$\frac{1}{2} \preceq x_1 \preceq \cdots \preceq x_n \preceq 1 \, .$$

Let $x_0 = \frac{1}{2}$ and $x_{n+1} = 1$, then by definition of $\preceq$, we have

$$[x_j, x_{j+1}] \subset B(x_j, \varepsilon_{x_j}) \cup B(x_{j+1}, \varepsilon_{x_{j+1}}) \subset O_{i_{x_j}} \cup O_{i_{x_{j+1}}}$$

for $j = 0, .., n$. Thus, we deduce

$$[\frac{1}{2}, 1] \subset \bigcup_{j=0}^{n} [x_j, x_{j+1}] \subset \bigcup_{j=0}^{n+1} O_{i_{x_j}} \, .$$

Doing the same trick with $[0, \frac{1}{2}]$, we find

$$[0, 1] \subset \bigcup_{j=0}^{m+1} O_{i_{x'_j}} \cup \bigcup_{j=0}^{n+1} O_{i_{x_j}}$$

and we have found our finite subcover. ■

PROPOSITION 3.4. *Let $B \subset X$ be closed set and $C \subset X$ be a compact set, then*

$$B \cap C$$

is compact

Proof: Let $B \cap C \subset \bigcup O_i$ be an open cover. then

$$C \subset (X \setminus B) \cup \bigcup_i O_i$$

is an open cover for C, hence we can find a finite subcover

$$C \subset (X \setminus B) \cup O_{i_1} \cup \cdots \cup O_{i_n} \, .$$

Thus

$$B \cap C \ \subset O_{i_1} \cup \cdots \cup O_{i_n}$$

is a finite subcover. ■

Lemma 3.5. *Let (X,d) be a metric space and $D \subset X$ be a countable dense set in X, then for every subset $C \subset X$ and every open cover*

$$C \subset \bigcup_i O_i$$

we can find a countable subcover of balls.

Proof: Let us enumerate D as $D = \{d_n \,|\, n \in \mathbb{N}\}$. Let $x \in C$ and find $i \in I$ and $\varepsilon > 0$ such that

$$x \in B(x,\varepsilon) \subset O_i \ .$$

Let $k > \frac{2}{\varepsilon}$. By density, we can find an $n \in \mathbb{N}$ such that

$$d(x,d_n) < \frac{1}{2k} \ .$$

Then

$$x \in B(d_n, \frac{1}{2k}) \subset B(x, \frac{1}{k}) \subset B(x,\varepsilon) \subset O_i \ .$$

Let us define

$$M \ = \ \{(n,k) \mid \exists_{i \in I} B(d_n, \frac{1}{2k}) \subset O_i\} \ .$$

Then $M \subset \mathbb{N}^2$ is countable and hence there exists a map $\phi : \mathbb{N} \to M$ which is surjective (=onto). Hence for $V_m = B(d_{\phi_1(m)}, \frac{1}{2\phi_2(m)})$, ϕ_1, ϕ_2 the 2 components of ϕ we have

$$C \subset \bigcup_m V_m$$

and (V_m) is a countable subcover of balls of the original cover (O_i). ∎

Theorem 3.6. *Let (X,d) be a metric space and D subset X be countable dense subset. Let $C \subset X$ be a subset. Then the following are equivalent*

- i) a) *Every Cauchy sequence of elements in C converges to a limit in C.*
  b) *For every $\varepsilon > 0$ there exists points $x_1, ..., x_n \in X$ such that*

$$C \subset B(x_1,\varepsilon) \cup \cdots \cup B(x,\varepsilon) \ .$$

- ii) *Every sequence in C has a convergent subsequence.*
- iii) *C is compact.*

Proof: *i)* $\Rightarrow$ *ii)*. Let (x_n) be a sequence. Inductively, we will construct infinite subset $A_1 \supset A_2 \supset A_3 \cdots$ and y_1, y_2, y_3,... in X such that

$$\forall_{l \in A_j} \ : \ d(x_l, y_j) < 2^{-j-1} \ .$$

Put $A_0 = \mathbb{N}$. Let us assume $A_1 \supset A_2 \supset \cdots A_n$ and $y_1, ..., y_n$ have been constructed. We put $\varepsilon = 2^{-n-2}$ and apply condition $i)b)$ to find $z_1, ..., z_m$ such that

$$C \ \subset \ B(z_1, \varepsilon) \cup \cdots \cup B(z_m, \varepsilon) \ .$$

We claim that there must be a $1 \leq k \leq m$ such that

$$A_n(k) \ = \ \{l \in A_n \,|\, x_n \in B(z_k, \varepsilon)\}$$

has infinitely many elements. Indeed, we have

$$A_n(1) \cup \cdots \cup A_n(m) \ = \ A_n \ .$$

If they were all finite, then a finite union of finite sets would have finitely many elements. However A_n is infinite. Contradiction! Thus, we can find a k with $A_n(k)$ infinite and put $A_{n+1} = A_n(k)$ and $y_{n+1} = z_k$. So the inductive procedure is finished. Now, we can find an increasing sequence (n_j) such that $n_j \in A_j$ and deduce

$$d(x_{n_j}, x_{n_{j+1}}) \ \leq \ d(x_{n_j}, y_j) + d(y_j, x_{n_{j+1}}) < \frac{1}{2}2^{-j} + \frac{1}{2}2^{-j} \ = \ 2^{-j}$$

because $n_j \in A_j$ and $n_{j+1} \in A_{j+1} \subset A_j$. Thus (x_{n_j}) is Cauchy. Indeed, be induction, we deduce for $j < m$ that

$$\begin{aligned} d(x_{n_j}, x_{n_m}) \ &\leq \ d(x_{n_j}, x_{n_{j+1}}) + d(x_{n_{j+1}}, x_{n_{j+2}}) \cdots d(x_{n_{m-1}}, x_{n_m}) \\ &\leq \ 2^{-j} \sum_{k=0}^{m-1} 2^{-k} \ = \ 2^{1-j} \ . \end{aligned}$$

This easily implies the Cauchy sequence condition. By a) it converges to some $x \in C$. We got our convergent subsequence.

$ii) \Rightarrow iii)$: By Lemma 3.5, we can assume that

$$C \subset \bigcup_k O_k$$

and O_k's open. If we can find an n such that

$$C \subset O_1 \cup \cdots \cup O_n$$

the assertion is proved. Assume that is not the case and choose for every $n \in \mathbb{N}$ an $x_n \in C \setminus O_1 \cup \cdots \cup O_n$. According to the assumption, we have a convergent subsequence, i.e. $\lim_k x_{n_k} = x \in C$. Then $x \in O_{n_0}$ for some n_0 and there exists a $\varepsilon > 0$ such that

$$B(x, \varepsilon) \ \subset O_{n_0} \ .$$

By convergence, we find a k_0 such that $d(x, x_{n_k}) < \varepsilon$ for all $k > k_0$. In particular, we find a $k > k_0$ such that $n_k > n_0$. Thus

$$x_{n_k} \in B(x, \varepsilon) \in O_{n_0} \subset O_1 \cup \cdots \cup O_{n_k} .$$

Contradicting the choice of the (x_n)'s. We are done.
iii) $\Rightarrow$ *i)b)* Let $\varepsilon > 0$ and then

$$C \subset \bigcup_{x \in C} B(x, \varepsilon) .$$

thus a finite subcover yields *b)*.
iii) $\Rightarrow$ *i)a)* Let (x_n) be a Cauchy sequence. Assume it is not converging to some element $x \in C$. This means

$$\forall x \in C \exists \varepsilon(x) > 0 \forall n_0 \exists n > n_0 \; d(x_n, x) > \varepsilon . \tag{3.1}$$

Then

$$C \subset \bigcup_{x \in C} B(x, \frac{\varepsilon(x)}{2}) .$$

Let

$$C \subset B(y_1, \frac{\varepsilon(y_1)}{2}) \cup \cdots \cup B(y_1, \frac{\varepsilon(y_1)}{2})$$

be a finite subcover (compactness). Then there exists at least one $1 \le k \le m$ such that

$$A_k = \{n \in \mathbb{N} \mid d(x_n, y_k) < \frac{\varepsilon(y_k)}{2}\}$$

is infinite. Fix that k and apply the Cauchy criterion to find n_0 such that

$$d(x_n, x_{n'}) < \frac{\varepsilon(y_k)}{2}$$

for all $n, n' > n_0$. By (3.1), we can find an $n > n_0$ such that

$$d(x_n, y_k) > \varepsilon(y_k) .$$

Since A_k is infinite, we can find an $n' > n_0$ in A_k thus

$$\begin{aligned} \varepsilon(y_k) &< d(x_{n'}, y_k) \le d(x_n, x_{n'}) + d(x_{n'}, y_k) \\ &< \frac{\varepsilon(y_k)}{2} + \frac{\varepsilon(y_k)}{2} = \varepsilon(y_k) . \end{aligned}$$

A contradiction. Thus the Cauchy sequence has to converge to some point in C. ∎

Corollary 3.7. *Every intervall* $[a, b] \subset \mathbb{R}$ *with* $a < b \in \mathbb{R}$ *is compact*

Proof: It is easy to see that $X \setminus [a,b]$ is open. Hence, by Proposition 3.1 $[a,b]$ is complete, i.e. i)a) is satisfied. Given $\varepsilon > 0$, we can find $k > \frac{1}{\varepsilon}$. For $m > k(b-a)$ we derive

$$[a,b] \subset \bigcup_{j=0}^{m} B(a + \frac{j}{k}, \varepsilon) .$$

Thus the Theorem 3.6 applies. ■

LEMMA 3.8. *Let $r > 0$ and $n \in \mathbb{N}$, the set $C_r = [-r,r]^n$ is compact.*

Proof: Let $x \notin C_r$, then there exists an index $j \in \{1,..,n\}$ such that $|x_j| > r$. Let $\varepsilon = |x_j| - r$ and $y \in \mathbb{R}^n$ such that

$$\max_{i=1,..,n} |x_i - y_i| < \varepsilon ,$$

then

$$|y_j| \;=\; |y_j - x_j + x_j| \;\geq\; |x_j| - |y_j - x_j| \;>\; |x_j| - \varepsilon \;=\; r .$$

thus $y \notin C_r$. Hence, C_r is closed and according to Proposition 1.6, we deduce that C_r is complete.
For $n = 1$ and $\varepsilon > 0$, we have seen above that for $k > \frac{1}{\varepsilon}$ and $m > \frac{2r}{k}$

$$[-r,r] \subset \bigcup_{j=0}^{m} B(-r + \frac{j}{k}, \varepsilon) .$$

Therefore

$$[-r,r]^n \subset \bigcup_{j_1,...,j_n=0,...m} B_\infty((-r + \frac{j_1}{k}, .., -r + \frac{j_n}{k}), \varepsilon) .$$

Thus i)a) and i)b) are satisfies and the Theorem 3.6 implies the assertion (The separable dense subset is $\mathbb{Q}^n$.) ■

THEOREM 3.9. *Let $C \subset \mathbb{R}^n$ be a subset. The following are equivalent*

1) *C is compact.*
2) *C is closed and there exists an r such that*

$$C \subset B(0,R) .$$

(That is C is bounded.)

Proof: 2) $\Rightarrow$ 1) Let

$$C \subset B(0,R) \subset [-R,R]^n$$

be a closed set. Since $[-R,R]^n$ is compact, we deduce from Proposition 3.4 that C is compact as well.

1) $\Rightarrow$ 2) Let C subset $\mathbb{R}^n$ be a compact set. According to Theorem 3.6 i)b), we find

$$C \subset B(x_1, 1) \cup \cdots \cup B(x_m, 1)$$

thus for $r = \max_{i=1,..,m}(d(x_i, 0) + 1)$ we have

$$C \subset B(0, r) \,.$$

Moreover, by Theorem 3.6 i)a) and Proposition 3.1, we deduce that C is closed. $\blacksquare$