CHAPTER 1

Analysis

1. Sequences and series

Definition 1.1. *A sequence (x_n) is called increasing if*

$$x_n \leq x_m$$

for all $n \leq m$. A sequence (x_n) is called decreasing if $(-x_n)$ is increasing. A sequence is called monotone if it is either increasing or decreasing.

Theorem 1.2. *Every sequence has a monotone subsequence.*

Lemma 1.3. *A bounded monotone sequence converges.*

Corollary 1.4. *A bounded sequence has a convergent subsequence*

Corollary 1.5. *A closed bouned intervall is compact.*

(Second Proof!).

Definition 1.6. $Lim(x_n) = \{x \,:\, \exists (n_k) x = \lim_k x_{n_k}\}$.

Proposition 1.7. $Lim(x_n) \;=\; \bigcup_m cl(\{x_n \; pln \geq m\})$ *is closed.*

Definition 1.8. *Let (x_n) be a bounded sequence. Then*

$$\limsup_n x_n \;=\; \sup Lim(x_n) \quad \textit{and} \quad \liminf_n x_n \;=\; \inf Lim(x_n)\;.$$

Project: Show that $x = \limsup_n x_n$ if

a) For every $\varepsilon > 0$ there exits n_0 such that

$$x_n \;=\; x + \varepsilon$$

for all $n > n_0$.

b) For every $\varepsilon > 0$ and $n > 0$ there exists $m > n$ such that

$$x - \varepsilon < x_m\;.$$

Definition 1.9. $\sum_n a_n$ *converges if* $s_n = \sum\limits_{k=1}^{n} a_k$ *converges.*

Example: If $|a| < 1$, then $\sum_n a^n$ converges.

Lemma 1.10. *If $\sum_n a_n$ converges then $\lim_n a_n = 0$.*

LEMMA 1.11. *Let (b_n) and (a_n) be sequences such that there exists $C > 0$ and n_0 such that*

$$|a_n| \le C|b_n|$$

for all $n > n_0$. If $\sum_n |b_n|$ converges, then

$$\sum_n a_n$$

converges.

Applications: Root-test, quotient test.