0.1 Borell-Cantelli Lemma

Proposition 0.1.1 *Borell-Cantelli (Part 1). Let (E_n) be events and*

$$E = \bigcap_{n_0} \bigcup_{n \geq n_0} E_n .$$

If

$$\sum_n P(E_n) < \infty$$

then

$$P(E) = 0 .$$

Observation: $P(\bigcup_n A_n) \leq \sum_n P(A_n)$

Proof: Define $B_1 = A_1$ and $B_n = A_n \cap (A_1 \cup \cdots \cup A_{n-1})^c$. Then then (B_n)'s are disjoint and $B_n \subset A_n$ and

$$\bigcup_n A_n = \bigcup_n B_n .$$

Thus

$$P(\bigcup_n A_n) = P(\bigcup_n B_n) = \sum_n P(B_n) \leq \sum_n P(A_n) .$$

Here we go. □

A sequence (E_i) is called independent if

$$P(E_{i_1} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \cdots P(E_{i_k})$$

for every k and mutually different indices $i_1, ..., i_k$.

Lemma 0.1.2 *If $E_1, ..., E_n$ are independent, then*

$$P(E_1 \cup \cdots \cup E_n) = 1 - \prod_{i=1}^{n} (1 - P(E_i)) .$$

HW

Lemma 0.1.3 *Let $0 < t < 1$, then*

$$-\log(1-t) \;\geq\; t\,.$$

Proof: By the fundamental theorem of calculus, we have

$$-\log(1-t) \;=\; \log(1) - \log(1-t) \;=\; \int_{1-t}^{1} \frac{dx}{x} \;\geq\; \int_{1-t}^{1} dx \;=\; 1-(1-t) \;=\; t\,.$$

Thats it. □

Proposition 0.1.4 *Borell-Cantelli (Part 2). Let (E_n) be independent events and*

$$E \;=\; \bigcap_{n_0} \bigcup_{n \geq n_0} E_n\,.$$

Then

$$\sum_n P(E_n) < \infty$$

if and only if

$$P(E) = 0\,.$$

Proof: We only have to show that $P(E) = 0$ implies that $\sum_{n=1}^{k} P(E_n)$ is a Cauchy sequence. Using the limit properties of probability measures, we deduce from Lemma 0.1.2

$$\begin{aligned} 0 = P(E) \;&=\; \lim_{n_0\to\infty} P(\bigcup_{n\geq n_0}) \;=\; \lim_{n_0\to\infty}\lim_{m\to\infty} P(\bigcup_{n=n_0}^{m} E_n) \\ &= \lim_{n_0\to\infty}\lim_{m\to\infty} 1 - \prod_{n=n_0}^{m} (1-P(E_n))\,. \end{aligned}$$

This $P(E) = 0$ implies

$$\lim_{n_0\to\infty}\lim_{m\to\infty} \;=\; \prod_{n=n_0}^{m} (1-P(E_n)) \;=\; 1\,.$$

By the continuity of the logarithm this implies

$$\lim_{n_0\to\infty}\lim_{m\to\infty} -\log(\prod_{n=n_0}^{m} (1-P(E_n))) \;=\; 0\,.$$

Now, we use the properties of the logarithm and find

$$-\log(\prod_{n=n_0}^{m}(1-P(E_n))) \;=\; \sum_{n=n_0}^{m} -\log(1-P(E_n)) \;\geq\; \sum_{n=n_0}^{m} P(E_n)\;.$$

Thus $P(E)=0$ implies

$$\lim_{n_0\to\infty}\lim_{m\to\infty}\sum_{n=n_0}^{m} P(E_n) \;=\; 0\;.$$

Thus $\sum_{n=1}^{k} P(E_n)$ is a Cauchy sequence and we are done. □