Warm up test-Probability 361

Problem 1: Let $0 < q < 1$. Show that

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} .$$

Problem 2: Show that $f(x) = x^2$ is differentiable and calculate

$$\int_1^{11} x dx .$$

Problem 3: How many different subsets are there in $\{0, 1, 2\}$?

Quiz 1- 361

Name:

1. Show that $p(E_1 \cup E_2) = P(E_1) + p(E_2) - p(E_1 \cap E_2)$ using the axioms of probability and their elementary consequences.

2. Let $A_1, ..., A_n$ be the boxes described in the lecture. Determine the conditional probability to get a second gold if you got a first one.

1. Hour exam-Math 361

Name:

1. Let $\Omega = \{1, ..., m\}$ and $P(A) = \frac{|A|}{m}$. Let $A_1, ..., A_r$ be r subset such that

$$\sum_{i=1}^{r} P(A_i) < 1 \, .$$

Show that there exists an element $k \in A_1^c \cap \cdots \cap A_r^c$. (Hint: Try $r = 2$).

2. (a) Use the 'super-cute' trick to show that

$$e^{-\lambda}\sum_{n=0}^{\infty} n^2 \frac{\lambda^n}{n!} \;=\; \lambda + \lambda^2$$

for all $\lambda > 0$.

(b) Let X be Poisson distributed with $P(X = n) = e^{-\lambda}\frac{\lambda^n}{n!}$. Calculate

$$EX^2 \,.$$

3. In a series of games a team wins, if it wins 3 out if 5. Let us assume the games are independent (ignoring psychology) and that the probability of team A to win is p.

   (a) Determine the formula for the probability that team A wins the series. (For $p = 1/2$ you should expect ?-use this to check whether you formula is correct.)

   (b) (extra credit) Let $B \subset \Omega$ be the event that team A wins. Define $X : B \to \{3, 4, 5\}$ to be the number of games used to win the series. Calculate

$$E(X|B) = \frac{\sum_{i=3}^{5} iP(X = i)}{P(B)}$$

   for $p = 1/2$. That is the expected number of games, given that team A wins. Even for arbitrary p this number should be between 3 and 5, why?

2. Hour exam-Math 361

Name:

1. (10P) Let X be a continuous random variable with density function f_X. Let $\alpha > 0$. Show that

$$f_{\alpha X}(x) \;=\; \frac{1}{\alpha} f_X(\frac{x}{\alpha})\,.$$

Solution: We have

$$F_{\alpha X}(a) \;=\; P(\alpha X \le a) \;=\; P(X \le a/\alpha) \;=\; \int\limits_{-\infty}^{a/\alpha} f_X(x)dx \;=\; \int\limits_{-\infty}^{a} f_X(x/\alpha)\frac{dx}{\alpha}\,.$$

Differentiation yields

$$f_{\alpha X}(a) \;=\; \frac{\partial}{\partial a} F_{\alpha X}(a) \;=\; \frac{f_X(a/\alpha)}{\alpha}\,.$$

2. (15P) Let U be uniformly distributed over $[0,1]$ and X be uniformly distributed over $[0,U]$. Calculate the distribution of U over X.

Solution: By definition $f_U(u) \;=\; 1$ for $0 \le u \le 1$ and 0 else. Moreover, by assumption

$$f_{X|U}(x|u) \;=\; \begin{cases} \frac{1}{u} & \text{if } 0 \le x \le u \\ 0 & \text{else} \end{cases}\,.$$

Therefore

$$f_{X,U}(x,u) \;=\; \frac{1}{u}$$

whenever $0 \le x \le u$. This yields

$$f_{U|X}(u|x) \;=\; \frac{f_{X,U}(x,u)}{f_X(x)}\,.$$

However, for $0 < x < 1$ we have

$$f_X(x) \;=\; \int_x^1 \frac{1}{u} du \;=\; \ln 1/x > 0\,.$$

This yields

$$f_{U|X}(u|x) \;=\; \frac{1}{-u \ln x}\,.$$

In other words U has the '$1/u$-distribution' over $[X,1]$ and X is itself $\ln(1/x)$-distributed (i.e. $f_X(x) = -\ln x$) on $[0,1]$.

3. (20P) Let X be exponential distributed with parameter 1 and Y be exponentially distributed with parameter X. Calculate

$$P(X < 1|Y = 1) = \int_0^1 f_{X|Y}(x|1)dx \,. \qquad (0.0.1)$$

Solution: In general if f_Z is the density function of random variable Z, then

$$P(Z \in A) = \int_A f_Z(x)dx \,.$$

(0.0.1) is the special case for the random variable $Z = E(X|Y = 1)$. In order to determine this distribution, we first determine the joint distribution function

$$f_{X,Y}(x, y) = f_{Y|X}(y|x)f_X(x) = xe^{-xy}e^{-x} = xe^{-x(y+1)} \,.$$

Then, we get

$$f_Y(y) = \int_0^\infty xe^{-x(y+1)}dx = \frac{xe^{-x(y+1)}}{-(y+1)}|_0^\infty + \int_0^\infty \frac{e^{-x(y+1)}}{y+1}dx = \frac{1}{(y+1)^2} \,.$$

And hence, we get

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = (y+1)^2 xe^{-x(y+1)} \,.$$

In particular for $y = 1$, this yields

$$P(X < 1|Y = 1) = \int_0^1 4xe^{-2x}dx = 4\left(\frac{xe^{-2x}}{-2}|_0^1 + \int_0^1 \frac{e^{-2x}}{2}dx\right)$$
$$= 4(-\frac{1}{2}e^{-2} - \frac{e^{-2}}{4} + \frac{1}{4}) = 1 - 3e^{-2} \,.$$

4. Given is an urn with m balls, a of them are green and $m - a$ of them are red. With probability a/m we replace a red ball by a green ball. With probability $(m - a)/m$ we don't do anything. We repeat this process (such that the new replacements are independent of the previous replacements) and denote N_a the minimal number of steps we need to turn all balls green. Find the expected value of N_a in particular for $a = 1$.

Solution 1: We really count all the steps. Let X be the random variable such that $X = 1$ if we replace red by green and $X = 0$ if we replace green by red.

$$E[N_a] \;=\; P(X=1)E(N_a|X=1) + P(X=0)E(N_a|X=0)\,.$$

In both cases we start a new game. If $X = 1$, we have $a+1$ green balls. If $X = 0$, we are as good of as before. In both cases we need an extra step to conclude. Thus, we get

$$E[N_a] \;=\; \frac{a}{m}(1+E(N_{a+1})) + \frac{m-a}{m}(1+E(N_a)) \;=\; 1 + \frac{a}{m}E(N_{a+1}) + \frac{m-a}{m}E(N_a)\,.$$

This yields

$$\frac{a}{m}E[N_a] \;=\; 1 + \frac{a}{m}E[N_{a+1}]$$

and hence

$$E[N_a] \;=\; \frac{m}{a} + E[N_{a+1}]\,.$$

For the case $a = m-1$, we get

$$E_[N_{m-1}] \;=\; \frac{m}{m-1}\,.$$

Thus for $a = 1$, we find

$$E[N_1] = \frac{m}{1} + E[N_2] \;=\; \frac{m}{1} + \frac{m}{2} + E[N_3] \;=\; \sum_{j=1}^{m} \frac{m}{j}$$

$$\cong m \int_1^m \frac{dx}{x} \;=\; m \ln m\,.$$

Actually there is a constant $\gamma > 0$ such that

$$\lim_{m\to\infty} \frac{\sum_{k=1}^{m} \frac{1}{k}}{\ln m} \;=\; \gamma$$

So we expect $\gamma m \ln m$ many steps to replace all green. For $a = 2$ and $m = 4$, we get

$$E[N_2] \;=\; \frac{4}{2} + E[N_3] \;=\; \frac{4}{2} + \frac{4}{3} \;=\; \frac{10}{3}\,.$$

Solution 2: We only count the steps M_a where we don't do anything. We use X as above. Then

$$E[M_a] \ = \ P(X=1)E(M_a|X=1) + P(X=0)E(M_a|X=0) \ .$$

However now, we have $E(M_a|X=1) = E[M_{a+1}]$ because we got a green ball and this action is not on our ticker. This yields

$$E[M_a] \ = \ \frac{a}{m}E[M_{a+1}] + \frac{m-a}{m}(1+E[M_a]) \ .$$

Now, we get

$$E[M_a] \ = \ \frac{m-a}{a} + E[M_{a+1}] \ .$$

For $a = m-1$, we find $E(M_a|X=1) = E[M_m] = 0$ and thus $E[M_{m=1}] = \frac{1}{m-1}$. In this case, we get

$$\begin{aligned} E[M_a] = \frac{m-a}{a} + E_{[}M_{a+1}] \ &= \ \frac{m-a}{a} + \frac{m-(a+1)}{a+1} + \cdots + \frac{1}{m-1} \\ = \sum_{k=a}^{m-1} \frac{m}{a} - (m-a) \ . \end{aligned}$$

Indeed,

$$E[M_a] + (m-a) \ = \ E[N_a]$$

because we have to turn the $(m-a)$ balls green. In that case we get

$$E[M_1] \ = \ m(\sum_{k=1}^{m-1} \frac{1}{k}) - m + 1 \cong \gamma m \ln m - m \ .$$

For $m = 8$ and $a = 6$, we get here

$$E[M_6] \ = \ \frac{8-6}{6} + \frac{8-7}{7} \ = \ \frac{20}{42} \ = \ \frac{10}{21} \ .$$

Final exam

Name:

I will take the maximum number of points in problem no 8 and problem no 9. (You can get an A if you skip one of them!). The problems with *give extra credit. Let us start with some warming-up questions.

1. (10P) Given an example of a probability space and a probability measure.

2. (10P) Given an example of two independent random variables on probability space including the sample space and the probablity measure. (If possible justify your claim)

3. (10P) What can you say about the distribution of the sum of two independent normal random variables?

4. (15P) Find a formula for the distribution function $f_{XY}(a)$ for $a > 0$ and two independent normal random variables X and Y.

5. (15P) Calculate the third moment of a λ-Poisson distribution using the moment generating function.

6. With probability p a coin turns up head. We continue flipping coins independently until head and tail have shown up for the first time.

   (a) (15P) What is the probability that the game ends with head? (Hint: condition on the first outcome).

   (b) (20P) Calculate the expected numbers Z_{total} of trials. What is the probability that the game ends with head. (Hint: First consider the game Z_{head} where we stop if head shows up. By conditioning on the first outcome show $E[Z_{head}] = \frac{1}{p}$. Similarly, $E[Z_{tail}] = \frac{1}{1-p}$. Condition once more).

7. (20P) Let X and Y be independent exponentially distributed random variables. Use conditional probability to calculate

$$P(X < Y)\,.$$

Extra credit: *Show that the answer is different for two independent Poisson distributed random variables with parameter λ.

8. We consider a gambler who on every play is equally likely to win or loose. This is modeled by a sequence (X_i) of independent, identically distributed random variables (X_i) such that
$$P(X_i = 1) \;=\; \frac{1}{2} \;=\; P(X_i = -1)\,.$$
The gamblers winning after n-steps is given by
$$S_n \;=\; \sum_{i=1}^{n} X_i\,.$$
Our goal is two show
$$P(S_n \leq a) \leq e^{-\frac{a^2}{2n}}\,.$$
   (a) (10P) Let X be a random variable with $P(X = 1) = \frac{1}{2} = P(X = -1)$. Calculate the moment generating function $M_X(t)$ and use the inequality
$$\frac{1}{2}(e^t + e^{-t}) \;\leq\; e^{\frac{t^2}{2}} \tag{0.0.2}$$
   to show
$$E[e^{tX}] \;\leq\; e^{\frac{t^2}{2}}\,.$$

(b) (15P) Let $S_n = \sum_{i=1}^{n} X_i$ as indicated by the problem. Show that

$$M_{S_n}(t) \leq e^{\frac{nt^2}{2}} .$$

(c) (15P) Use the Chernoff bound $P(S \geq a) \leq e^{-ta} M_S(t)$ and show

$$P(S_n \geq a) \leq e^{-\frac{a^2}{2n}} .$$

(d) (10P) Show that

$$P(S_{100} \geq 20) \leq e^{-2} \approx 0.135 .$$

(e) (15P) What do you obtain if you apply the central limit theorem and the Chernoff bound ($P(X \geq a) \leq e^{\frac{a^2}{2}}$ for the normal distribution.

9. The proof of the strong law of large numbers shows that for the n-sum $S_n = \sum_{i=1}^{n} X_i$ of independent identically distributed random variables (X_i) and $X = X_1$, we have

$$E[S_n^4] \;\le\; E[X^4](\frac{1}{n^3} + \frac{3}{n^2})$$

for all $n \in \mathbb{N}$. This implies

$$\sum_{n=1}^{\infty} \frac{E[S_n^4]}{n^4} \;\le\; \frac{15}{2} E[X^4] \,. \qquad (0.0.3)$$

Now, we consider a stock market with a sequence (X_i) of identically distributed normal $N(0,1)$ distributed random variables and the corresponding

$$Y_n \;=\; Y_{n-1} + X_n \,.$$

In the following you may use

$$E[X^4] \;\le\; 2$$

for a N(0,1) distributed random variable X.

(a) (20P) Use the Markov inequality and show that with probability at least $\frac{15}{16}$ we have

$$|Y_n - Y_0| \;\le\; 2n$$

for all $n \in \mathbb{N}$.

(b) (15P) Now, we don't care what happens in the first 10 days. Note that we have

$$\sum_{n>10} n^{-3} \leq 10^{-3} + \int_{10}^{\infty} x^{-3} dx \;\leq\; 10^{-3} + \frac{1}{2} \times 10^{-2} \;=\; 6 \times 10^{-3} \,.$$

and

$$\sum_{n>10} n^{-2} \leq 10^{-2} + \int_{10}^{\infty} x^{-2} dx \;=\; 10^{-2} + 10^{-1} \;=\; 11 \times 10^{-2} \,.$$

Give an estimate for the probability such that for all $n > 10$, we have

$$|Y_n - Y_0| \;\leq\; 2n \,.$$

(c) (10P) What do you observe?

2. Hour exam-Math 361

Name:

1. Let X be a continuous random variable with denisty f_X. Let $\alpha > 0$. Find the density $f_{\alpha+X}$.

2. Let U be uniformly distributed over $[0, 1]$ and X be exponential distributed over $[0, U]$. Find the distribution of U over X.

3. The joint density of X and Y is given by

$$f_{XY}(x,y) \;=\; ce^{\frac{1}{2}(x^2+2axy+y^2)}\;.$$

Find c and show that X and Y are independent if and only if $a = 0$.

4. We have a green ball and $m - a$ red balls. We toss a coin at random. With probability a/m the coin ends up 1. With probability $m - a/m$ the coin ends up 0. If 1 appears we replace red by a green ball. If 0 we don't do anything. Use conditional probability to determine the expected number tossing's we need to find all balls green if $m = 6$ and $a = 3$. Extra credit is for find a formula for general m and $a = 1$.