Crash course on integration

Let $(\Omega, \mathcal{F}, p)$ be a probability space. A rv $f : \Omega \to \mathbb{R}$ is called simple function ($f \in S$) if there exists $A_1, ..., A_n$ and $r_1, .., r_n$ such that

$$f(\omega) \; = \; \sum_{i=1}^{n} r_i 1_{A_i}(\omega) \; .$$

Here

$$1_A(\omega) \; = \; \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{else} \end{cases} \; .$$

For a simple function

$$Ef \; = \; \int f dp \; = \; \sum_{i=1}^{n} r_i p(A_i) \; .$$

Lemma 0.0.1 *Let $f, g \in S$, then*

$$\int (f + tg) dp \; = \; \int f dp + t \int dgp \; .$$

Proof: Assume $f = \sum_{i=1}^{n} r_i 1_{A_i}$, $g = \sum_{j=1}^{n} s_j 1_{B_j}$, then

$$f + tg \; = \; \sum_{i=1}^{n} r_i 1_{A_i} + \sum_{j=1}^{n} t s_j 1_{B_j}$$

and thus

$$E(f + tg) \; = \; \sum_{i=1}^{n} r_i p(A_i) + t \sum_{j} s_j 1_{B_j} \; = \; E(f) + tE(g) \; .$$

The real problem is to show that $\int f dp$ is well-defined, but that's not your problem. $\square$

Proposition 0.0.2 *The map $\int : S \to \mathbb{R}$ has the floowing properties*

1. $\int (f + tg) dp \; = \; \int f dp + t \int g dp$,
2. $\left| \int f dp \right| \; \leq \; \int |f| dp$,
3. $\min(f) p(\Omega) \; \leq \; \int f dp \; \leq \; \max(f) p(\Omega)$.

Now, we want to define the integral for arbitrary rv's. We assume that $f : \Omega \to \mathbb{R}$ is measurable. Let $\varepsilon > 0$ and for $k \in \mathbb{Z}$ we define

$$A_{k,\varepsilon} \;=\; \{\omega \in \Omega \;:\; k\varepsilon < f(\omega) \;\le\; (k+1)\varepsilon\} \in \mathcal{F}$$

We say that f is *integrable* if

$$\sum_{k \in \mathbb{Z}} \max\{1, |k\varepsilon|\} p(A_{k,\varepsilon})$$

is finite for all ε. In that case, we define

$$\int f dp \;=\; \lim_{\varepsilon \to 0} \sum_{k \in \mathbb{Z}} k\varepsilon p(A_{k,\varepsilon}) \,.$$

Remark 0.0.3 **1)** *For simple functions this gives the same value.*
2) *(For experts) For every integrable functions there exists a sequence* (f_n) *of simple functions such that*

$$\lim_n \int |f_n - f| dp \;=\; 0$$

and

$$\int f dp \;=\; \lim_n \int f_n dp \,.$$

3) *Let* $\mathcal{B}$ *the smallest* σ*-algebra generated by intervals in* $\mathbb{R}$*. Then there exists a measure* $\mu : \mathcal{B} \to [0, \infty]$ *such that*

$$\mu(I) \;=\; |I|$$

holds for all intervals. The same definition replacing p *with* μ *applies to measurable functions* $f : \mathbb{R} \to \mathbb{R}$*.*

Let us show **1)** for convenience. Let $f : \Omega \to \mathbb{R}$ a simple function. Then $f(\Omega) = \{a_1, ..., a_n\}$ has only finitely many values. Define $B_i = \{\omega \;:\; f(\omega) = a_i\}$. Let $\varepsilon_0 \;=\; \min_{i \neq j} |a_i - a_j|$. If $\varepsilon < \varepsilon_0$, then every a_i belongs to at most one interval $(k\varepsilon, (k+1)\varepsilon)$. Therefore, we have exactly n elements $k_1, ..., k_n \in \mathbb{Z}$ such that

$$A_{k_i,\varepsilon} \;=\; B_i$$

for all $i = 1, ..., n$. This implies

$$\left| \sum_{k \in \mathbb{Z}} k\varepsilon p(A_{k,\varepsilon}) - \sum_{i=1}^{n} a_i p(B_i) \right| = \left| \sum_{i=1}^{n} (k_i \varepsilon - a_i) p(B_i) \right|$$
$$\leq \sum_{i=1}^{n} (a_i - k_i \varepsilon) p(B_i)$$
$$\leq \varepsilon \sum_{i=1} p(B_i) \ \leq \ \varepsilon \ .$$

Of course, we get

$$\lim_{\varepsilon \to 0} \sum_{k} k\varepsilon p(A_{k\varepsilon}) \ = \ \sum_{i=1}^{n} a_i p(A_i) \ = \ E(f) \ .$$

In general, we have the following theorem.

Theorem 0.0.4 *Let $\mathcal{L}^1(\Omega, p)$ the set of integrable functions. Let $f, g \in \mathcal{L}^1(\Omega, p)$. Then*

1. $f + tg \in L^1(\Omega, p)$ *and*

$$\int (f + tg) dp \ = \ \int f dp + t \int g dp,$$

2. $|\int f dp| \ \leq \ \int |f| dp,$

3. $\min(f) p(\Omega) \ \leq \ \int f dp \ \leq \ \max(f) p(\Omega),$

4. *Let $h \in L^1(\Omega, p)$ be positive, f and (f_n) rv's such that $|f_n| \leq h$ for all $n \in \mathbb{N}$ and there exists a set $A \subset \Omega$ such that $p(A) = 0$*

$$f(\omega) \ = \ \lim_{n} f_n(\omega)$$

for all $\omega \in A^c$. Then

$$\lim_{n} \int f_n dp \ = \ \int f dp \ .$$

One of the important properties is that continuous function on compact intervals are integrable and

$$\int_0^1 f(x) dx \ = \ \int f dp \ = \ E(f) \ .$$