

1. UNIT: AXIOMS AND ELEMENTARY PROPERTIES

- Goal
- What is probability?
- What is a probability space? (mathematical model)

1.1. Axioms of probability.

1.1.1. Set of outcomes, sample space.

Examples:

1.1.2. collection of events:

Notation (set theory): $EF = E \cap F$, $E \cup F = (E^c F^c)^c$, $\bigcup_n E_n$.

$\Sigma \subset 2^\Omega$ is called an algebra if

- (1) $\Omega \in \Sigma$.
- (2) $E \in \Sigma$ implies $E^c \in \Sigma$.
- (3) $E, F \in \Sigma$ implies $EF \in \Sigma$.

Σ is called a σ -algebra if E_1, E_2, \dots all in Σ implies

$$\bigcup_n E_n \in \Sigma .$$

(Why? later important for continuous random variables)

Examples:

1.1.3. *Probability measure.* A probability measure P is a function $P : \Sigma \rightarrow [0, 1]$ such that

$$P(\Omega) = 1$$

and for a countable collection of disjoint sets (E_n) in Σ we have

$$P\left(\bigcup_n E_n\right) = \sum_n P(E_n) .$$

Question 1: Find the definition of convergence in one of your math books.

Examples: $\Omega = \{1, \dots, m\}$ with $P(A) = \frac{|A|}{m}$. Product spaces, different weights, ordered/unordered pairs, binomial coefficients, card decks, rolling a dice, urn model, more combinatorics.

1.1.4. *Elementary properties and not so elementary properties.***Proposition 1.1.** (1) $P(\Omega) = 1$ (2) $P(E_1 \cup \dots \cup E_m) = \sum_{j=1}^m P(E_j)$. = holds for mutually disjoint sets.(3) $E \subset F$ implies $P(E) \leq P(F)$.(4) If $E_1 \subset E_2 \subset \dots$, then

$$\lim_n P(E_n) = P\left(\bigcup_n E_n\right).$$

(5) If $E_1 \supset E_2 \supset \dots$, then

$$\lim_n P(E_n) = P\left(\bigcap_n E_n\right).$$

Remark 1.2. (4) and (5) seem theoretical now, but are crucial if we want to talk about limit behaviour.**Examples:**1.2. **Conditional probability.****Theory:** If $P(F) > 0$, we define

$$P(E|F) = \frac{P(EF)}{P(F)}.$$

Then $P(\cdot|F)$ defines a new probability measure on Σ .**Remark:** $P(EF) = P(E|F)P(F)$.**Bayes formula:**

$$P(E) = P(E \cap F) + P(E \cap F^c) = P(E|F)P(F) + P(E|F^c)P(F^c).$$

(Mathematical simply, but tricky because F must be chosen according to the problem).**Example 1)** Three boxes with 2 pieces in it: 2 gold, 2 silver, one box gold and silver. What is the probability of a second gold given a first gold?**Example 2)** The king comes from a family of two children. What is the probability of having a sister.**Example 3)** No 3a,b, 3j, p=78.**Example 4)** No 10 (p=104).

1.2.1. *Independence.* Two events are independent if

$$P(EF) = P(E)P(F).$$

Proposition 1.3. *E and F independent iff E^c and F^c independent.*

Project: Odd's ratio.

Independence comes from independent coordinates

Proposition 1.4. *If E and F are independent, then there is a probability measure \tilde{P} on $\{(0,0), (0,1), (1,0), (1,1)\}$ such that*

$$P(E) = P(\{(0,0), (0,1)\}), P(F) = P(\{(0,0), (1,0)\}), P(EF) = P(\{(0,0)\}).$$

Example: parallel circuits

Example: No 62 b) $p=1/11$.

Problem: Player A plays against the bank. When head shows up he wins a dollar when tail shows up he loses a dollar. He starts with capital N and the game ends if he has lost the money. What is the probability for the end to stop.

The trick: conditional probability and recursion

Example: The gambler's ruin problem.

Example: more points to win (example 4i).

Example: n successes before m failures (example 5b).

Example: Laplace's rule of succession.